Some model-theoretic aspects of valued Frobenius fields

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§1 Immediate Expansions

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Remark

The true cases have a lot of applications in other areas, e.g. Diophantine geometry. Investigations on various restricted versions of Zilber's Trichotomy conjecture are still being carried out.

Zilber's Restricted Trichotomy Conj.(2012, open)

"assuming a strongly minimal structure M is interpretable in an algebraically closed field K and M is not locally modular, a field isomorphic to K is interpretable in M."

Definition

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$$(\forall x \in K^{\times}) \left[(x \in V) \bigvee \left(x^{-1} \in V \right) \right]$$

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$\mathscr{L}_{\mathrm{div}} := \mathscr{L}_{\mathrm{r}} \cup \{|\}$ denotes the **language of valued fields** (or valued rings). On a valued field, the division predicate is interpreted as

$$x \mid y \Leftrightarrow \frac{y}{x} \in V.$$

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Convention

Definable: always presumes parameters from the underlying universe.

Haskell-Macpherson(1998)

(1). Assume that M is an $\mathscr{L}_{\mathrm{div}}$-definable set in a valued algebraically closed field (K, V) and M is not $\mathscr{L}_{\mathrm{r}}$-definable over K, then V is $\mathscr{L}_{\mathrm{r}} \cup \{P_M\}$-definable over K.

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(2). Assume that M is an $\mathscr{L}_{\mathrm{div}}$-definable set in a convexly valued real closed field (K, V) and M is not $\mathscr{L}_{\mathrm{r}}$-definable over K, then V is $\mathscr{L}_{\mathrm{r}} \cup \{P_M\}$-definable over K.

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M a set, on which there is an $\mathscr{L}_1$-structure, denoted by $\mathcal{M}_1$, and an $\mathscr{L}_2$-structure, denoted by $\mathcal{M}_2$.

Definition

$\mathcal{M}_2$ is an **expansion** of $\mathcal{M}_1$, and $\mathcal{M}_1$ a **reduct** of $\mathcal{M}_2$, denoted by $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2$, if every $\mathscr{L}_1$-definable set over M is already $\mathscr{L}_2$-definable over M.

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Definition

Suppose that $\mathscr{L}$ is another first-order language and on M there is an $\mathscr{L}$-structure $\mathcal{M}$, such that $\mathcal{M}_1 \sqsubseteq \mathcal{M} \sqsubseteq \mathcal{M}_2$, then $\mathcal{M}$ is called an **intermediate (first-order) structure** between $\mathcal{M}_1$ and $\mathcal{M}_2$, or, of the pair $(\mathcal{M}_1, \mathcal{M}_2)$.

Definition

The two structures $\mathcal{M}_1$ and $\mathcal{M}_2$ are said to **have the same (first-order) structure**, denoted by $\mathcal{M}_1 \approx \mathcal{M}_2$, if $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2 \sqsubseteq \mathcal{M}_1$. Its negation is denoted by $\mathcal{M}_1 \not\approx \mathcal{M}_2$.

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$\mathcal{M}_2$ is a **proper expansion** of $\mathcal{M}_1$, and $\mathcal{M}_1$ a **proper reduct** of $\mathcal{M}_2$, if $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2 \not\approx \mathcal{M}_1$. An intermediate structure $\mathcal{M}$ between $\mathcal{M}_1$ and $\mathcal{M}_2$ is called a **proper intermediate structure** if $\mathcal{M}_1 \not\approx \mathcal{M} \not\approx \mathcal{M}_2$.

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$\mathcal{M}_2$ is an **immediate expansion** of $\mathcal{M}_1$, and $\mathcal{M}_1$ an **immediate reduct** of $\mathcal{M}_2$, denoted by $\mathcal{M}_1 ⧖ \mathcal{M}_2$ or $\mathcal{M}_2 ⧖ \mathcal{M}_1$, if $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2$ and there is no proper intermediate structures between $\mathcal{M}_1$ and $\mathcal{M}_2$. The negation of $\mathcal{M}_2 ⧖ \mathcal{M}_1$ is denoted by $\mathcal{M}_2 ⧗ \mathcal{M}_1$.

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Notation

If T_2 is a theory extending T_1, then by $T_2 \gtrdot T_1$, we mean that for any $\mathcal{M}_2 \models T_2$, if its reduct is denoted by $\mathcal{M}_1 \models T_1$ then we have $\mathcal{M}_2 \gtrdot \mathcal{M}_1$.

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Question

What valued field is an immediate expansion of its underlying field?

Known results

(1). ACVF ⧖ ACF.

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(3). SCVF ⧖ SCF, and the same scenario happens for some valued o-minimal fields.

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Remark

A family of examples of valued fields, $K(x_i, i \in \omega + \omega)$, which are non-immediate expansions of their underlying fields were given by Delon(2012).

Question

PACVF ⧖ PACF?

- PACVF is the $\mathscr{L}_{\mathrm{div}}$-theory of pseudo-algebraically closed nontrivially valued fields.
- PACF is the $\mathscr{L}_{\mathrm{r}}$-theory of pseudo-algebraically closed fields.

§2 Frobenius Fields

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(1). Separably closed fields are PAC.

(2). (Ax) Pseudo-finite fields (i.e. infinite models of the theory of all finite fields) are PAC.

(3). (Jarden, The PAC Nullstellensatz) For almost all finite tuples of the absolute Galois group of a countable Hilbertian field, their fixed fields are PAC.

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(7). (Ax, projectivity) If K is PAC and A, B are profinite groups, then for any epimorphisms $\rho : \mathrm{Gal}(K) \to A$ and $\alpha : B \to A$, there exists a homomorphism $\gamma : \mathrm{Gal}(K) \to B$ such that $\rho = \alpha \circ \gamma$.

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(6). (Haran-Lubotzki) The first-order theory of Frobenius fields is decidable.

Definition

A PAC field K is called a **Frobenius field** if $\mathrm{Gal}(K)$ satisfies the Embedding Property: for any profinite group epimorphisms $\zeta : \mathrm{Gal}(K) \to A$ and $\alpha : B \to A$ with B a finite quotient of $\mathrm{Gal}(K)$, there exists an epimorphism $\gamma : G \to B$ such that $\zeta = \alpha \circ \gamma$.

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Remark

These are both first-order in $\mathscr{L}_{\mathrm{r}}$.

§3 Quantifier Elimination

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Theorem(Cherlin-van den Dries-MacIntyre)

The theory of perfect Frobenius fields in the Galois formalism admits quantifier elimination.

The proof relies on the following important Embedding Lemma.

The Embedding Lemma(Jarden-Kiehne)

Let E/L and F/M be separable field extensions satisfying: E is countable and F is PAC and $\aleph_1$-saturated. Suppose that there are an isomorphism $\Phi_0 : L_s \to M_s$ with $\Phi_0(L) = M$ and a commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(E) & \xleftarrow{\ \varphi\ } & \mathrm{Gal}(F) \\ \Big\downarrow{\scriptstyle \mathrm{res}} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{Gal}(L) & \xleftarrow{\ \varphi_0\ } & \mathrm{Gal}(M) \end{array}$$

where φ_0 is the isomorphism induced by Φ_0 and φ is a homomorphism. If $\mathrm{char}(L) = p > 0$, then suppose furthermore that $[E : E^p] \leq [F : F^p]$.

Then there exists an extension of Φ_0 to an embedding $\Phi : E^{\mathrm{sep}} \to F^{\mathrm{sep}}$ which induces φ with $F/\Phi(E)$ separable.

Theorem-definition

Suppose that K is a field with $\mathrm{char}(K) = p > 0$. Then there exists a unique $e \in \omega \cup \{\infty\}$ such that $[K : K^p] = p^e$. This e is called the **exponent of imperfection** of K and $[K : K^p]$ the **degree of imperfection**.

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A set of elements $b_0, \ldots, b_{n-1} \in K$ are **p-independent** in K if all the p-monomials in $b_0, \ldots, b_{n-1}$ of the form

$$\prod_{j=0}^{n-1} b_j^{i(j)}, \quad i : n \to p$$

are linearly independent over K^p.

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A **p-basis** is a set of maximally p-independent elements in K.

$[K : K^p] = p^e$ iff K has a p-basis with exactly e elements.

Definition

A field extension L/K is **separable** if the p-independence relation is preserved.

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A field extension L/K is **separable** if the p-independence relation is preserved. If furthermore, K is relatively algebraically closed in L, then L/K is said to be **regular**.

Definition(Srour, 1986)

Suppose that K is a field of positive characteristic p. When $n > 0$ and $0 \leq i \leq p^n$, we define the **relative p-coordinate functions** $\lambda_{n,i}(x; y_1, \ldots, y_n)$ as follows: if $y_1, \ldots, y_n$ are not p-independent in K or $x \notin K^p(y_1, \ldots, y_n)$, then $\lambda_{n,i}(x; y_1, \ldots, y_n) = 0$, otherwise $\lambda_{n,i}(x; y_1, \ldots, y_n)$ is the p-th root of the unique i-th coefficient of x with respect $y_1, \ldots, y_n$ when x is written as a linear combination of p-monomials in $y_1, \ldots, y_n$ over K^p. If $n = 0$, then we defined $\lambda_{n,i}(x) = \lambda_{0,0}(x)$ to be $x^{1/p}$ if $x \in K^p$, 0 if $x \notin K^p$.

Definition

Given $n+1$ elements $a_0, \ldots, a_{n-1}, a_n$ in a field K, denote that the maximal splitting factor of

$$a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$$

over K is

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$$b_n X^n + b_{n-1} X^{n-1} + \cdots + b_1 X + b_0,$$

where $b_n, \ldots, b_0 \in K$, some of which could be zeros. For each $0 \leq i \leq n$, we define the i-th **splitting coefficient** to be

$$\theta_{n,i}(a_0, a_1, \ldots, a_n) := b_i.$$

Notation

$\mathscr{L}_{\theta,p,\mathrm{div}} := \mathscr{L}_{\mathrm{div}} \cup \{\lambda_{n,i}\}_{n,i} \cup \{\theta_j\}_j$.

Theorem(FrobVF$_{p^e}$ has QE)

For any $e \in \omega \cup \{\infty\}$, let FrobVF$_{p^e}$ be the theory of Frobenius non-trivially valued fields of characteristic p with exponent of imperfection e in

$$\mathscr{L}_{\theta,p,\mathrm{div}} \cup \{I_G\}_{G \text{ a finite group}},$$

where

$$K \models I_G \Leftrightarrow \text{“}\exists L/K \text{ such that } \mathrm{Gal}(L/K) \cong G\text{”}.$$

Then FrobVF$_{p^e}$ has quantifier elimination.

Valuation Theoretic Embedding Lemma

Suppose that valued fields (E, V_E) and (F, V_F) extend (L, V_L) and (M, V_M) resp. with E/L and F/M regular field extensions, that F is PAC and $|E|^+$-saturated in $\mathscr{L}_{\mathrm{div}}$ with the exponent of imperfection of E not more than that of F, and that there is a field isomorphism $\Phi_0 : L^{\mathrm{sep}} \to M^{\mathrm{sep}}$ and a commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(E) & \xleftarrow{\ \varphi\ } & \mathrm{Gal}(F) \\ {\scriptstyle \mathrm{res}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{Gal}(L) & \xleftarrow[\ \varphi_0\ ]{} & \mathrm{Gal}(M), \end{array}$$

where φ_0 is induced by Φ_0, φ is a homomorphism, and Φ_0 restricts to an $\mathscr{L}_{\mathrm{div}}$-isomorphism from (L, V_L) onto (M, V_M). Then there exists an extension of Φ_0 to a field embedding $\Phi : E^{\mathrm{sep}} \to F^{\mathrm{sep}}$ that induces φ with $F/\Phi(E)$ separable and that Φ restricts to an $\mathscr{L}_{\mathrm{div}}$-embedding from (E, V_E) into (F, V_F). If furthermore φ is surjective, then $F/\Phi(E)$ is regular.

Corollary(PACVF$^{\omega}_{p^e}$ **has QE)**

For any $e \in \omega \cup \{\infty\}$, the theory of ω-free pseudo-algebraically closed non-trivially valued fields of characteristic p with exponent of imperfection e in $\mathscr{L}_{\theta,p,\mathrm{div}}$, denoted by $\mathrm{PACVF}^{\omega}_{p^e}$, admits quantifier elimination.

§3 The Denseness Property

Kollár's Density Theorem(Kollár, 2007)

Suppose that K is a PAC field, V a non-trivial valuation on K^{alg}. Then for any geometrically integral K-variety X, $X(K)$ is V-dense in $X(K^{\mathrm{alg}})$.

Theorem(Bary-Soroker, 2012)

Let K be a PAC field, Y an absolutely irreducible smooth K-variety with ring of regular functions R, $f(X) \in R[X]$ a separable monic polynomial, and P a partition of $\deg(f)$. Assume that the induced embedding problem has a solution whose orbit type is P. Then there exists a Zariski dense set $\mathfrak{p} \in Y(K)$ such that $\varphi_{\mathfrak{p}}(f)$ is a separable polynomial of factorization type P.

Corollary

Let K be a pseudo-algebraically closed field, W a valuation ring on K^{alg}, Y a geometrically integral smooth K-variety with its ring of regular functions R, $f(X) \in R[X]$ a separable monic polynomial, P a partition of $\deg f$. Assume that the induced embedding problem has a solution whose orbit type is P. Then there exists a W-dense subset $\mathfrak{p} \in Y(K)$ of $Y(K^{\mathrm{alg}})$ such that $\varphi_{\mathfrak{p}}(f)$ is a separable polynomial of factorization type P.

Theorem

Suppose that $(K, V) \models \mathrm{PACVF}^{\omega}_{p^e}$ for a natural number $e < \infty$ and that W is a valuation on K^{alg} extending V. Then every $\mathscr{L}_{\theta,p,\mathrm{div}}$-definable set in K is W-dense in an $\mathscr{L}_{\mathrm{div}}$-definable set in K^{alg} with parameters from K. More precisely, if $\varphi(\boldsymbol{x}, \boldsymbol{a})$ is an $\mathscr{L}_{\theta,p,\mathrm{div}}$-formula with $\boldsymbol{a} \in K^n$, then there exists an $\mathscr{L}_{\mathrm{div}}$-formula $\tilde{\varphi}(\boldsymbol{x}, \boldsymbol{b})$ with $\boldsymbol{b} \in K^m$ such that $\varphi(K)$ is a W-dense subset of $\tilde{\varphi}\left(K^{\mathrm{alg}}\right)$.

To be proved recently

$\mathrm{PACVF}^{\omega}_{p^e} \mathrel{⧖} \mathrm{PACF}^{\omega}_{p^e}$.

Thank you!