A Choice-Free Cardinal Equality

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Introduction

For a (non-well-ordered) cardinal $\mathfrak{a}$, let

$$\mathrm{fin}(\mathfrak{a}) = \left|\{y \mid y \text{ is a finite subset of } x\}\right|,$$

where $|x| = \mathfrak{a}$; namely, $\mathrm{fin}(\mathfrak{a}) = [\mathfrak{a}]^{<\omega}$.

Theorem (Hessenberg 1906)

For all infinite well-ordered cardinals κ, $\kappa \cdot \kappa = \kappa$.

Corollary

For all infinite well-ordered cardinals κ, $\mathrm{fin}(\kappa) = \kappa$.

The *axiom of choice* trivializes the investigation of $\mathrm{fin}(\mathfrak{a})$.

Introduction

On the other hand, in the ordered Mostowski model, the cardinality $\mathfrak{a}$ of the set of atoms satisfies

$$\begin{aligned} \mathrm{fin}(\mathfrak{a}) < [\mathrm{fin}(\mathfrak{a})]^2 < \mathrm{fin}(\mathfrak{a})^2 < [\mathrm{fin}(\mathfrak{a})]^3 < \mathrm{fin}(\mathfrak{a})^3 < \cdots \\ < \mathrm{fin}(\mathrm{fin}(\mathfrak{a})) < \mathrm{fin}(\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))) < \cdots < \aleph_0 \cdot \mathrm{fin}(\mathfrak{a}). \end{aligned} \tag{1}$$

It is natural to ask which relationships between the *powers* of the cardinals in (1) for an arbitrary infinite cardinal $\mathfrak{a}$ can be proved without the aid of the axiom of choice.

Introduction

The first result of this kind is Läuchli's lemma, which states that, for all infinite cardinals $\mathfrak{a}$,

$$2^{\aleph_0 \cdot \mathrm{fin}(\mathfrak{a})} = 2^{\mathrm{fin}(\mathfrak{a})}.$$

Läuchli's lemma implies that, in the ordered Mostowski model, the powers of the cardinals in (1) are all equal, where $\mathfrak{a}$ is the cardinality of the set of atoms.

Introduction

In this talk, we first prove that the existence of an infinite cardinals $\mathfrak{a}$ such that

$$2^{\mathrm{fin}(\mathfrak{a})} < 2^{\mathrm{fin}(\mathfrak{a})^2} < 2^{\mathrm{fin}(\mathfrak{a})^3} < \cdots < 2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))}$$

is consistent with ZF, then we prove in ZF that, for all infinite cardinals $\mathfrak{a}$,

$$2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))} = 2^{\mathrm{fin}(\mathrm{fin}(\mathrm{fin}(\mathfrak{a})))} = 2^{\mathrm{fin}(\mathrm{fin}(\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))))} = \cdots$$

and

$$2^{\mathrm{fin}(\mathfrak{a})^n} = 2^{[\mathrm{fin}(\mathfrak{a})]^n},$$

which gives an answer to the above question.

Basic Fraenkel Model

Now we establish some consistency results by the method of permutation models. Permutation models are not models of ZF; they are models of ZFA (the Zermelo–Fraenkel set theory with atoms). Nevertheless, they indirectly give, via the Jech–Sochor theorem, models of ZF.

For our purpose, we only consider the basic Fraenkel model $\mathcal{V}_{\mathrm{F}}$. The set A of atoms of $\mathcal{V}_{\mathrm{F}}$ is denumerable, and $x \in \mathcal{V}_{\mathrm{F}}$ if and only if $x \subseteq \mathcal{V}_{\mathrm{F}}$ and x has a *finite support*, that is, a set $B \in \mathrm{fin}(A)$ such that every permutation of A fixing B pointwise also fixes x.

Basic Fraenkel Model

Theorem

Let A be the set of atoms of $\mathcal{V}_{\mathrm{F}}$ and let $\mathfrak{a} = |A|$. In $\mathcal{V}_{\mathrm{F}}$,

$$2^{\mathrm{fin}(\mathfrak{a})} < 2^{\mathrm{fin}(\mathfrak{a})^2} < 2^{\mathrm{fin}(\mathfrak{a})^3} < \cdots < 2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))}.$$

Proof. We begin with a definition:

$$\mathfrak{a}^{\underline{\mathfrak{b}}} = \left|\{f \mid f \text{ is an injection of } y \text{ into } x\}\right|$$

where $|x| = \mathfrak{a}$ and $|y| = \mathfrak{b}$.

Let $n \in \omega$. We prove that, in $\mathcal{V}_{\mathrm{F}}$,

$$\mathfrak{a}^{\underline{2^n}} \leqslant \mathrm{fin}(\mathfrak{a})^{n+1} \tag{1}$$

$$2^{\mathfrak{a}^{\underline{2^n}}} \nleqslant 2^{\mathrm{fin}(\mathfrak{a})^n}, \tag{2}$$

and hence the theorem follows.

Basic Fraenkel Model

Lemma

For all cardinals $\mathfrak{a}$ and all natural numbers n, $\mathfrak{a}^{\underline{2^n}} \leqslant \mathrm{fin}(\mathfrak{a})^{n+1}$.

Proof. Let f be the function on $x^{\underline{\wp(n)}}$ such that, for all $t \in x^{\underline{\wp(n)}}$, $f(t)$ is the function on $n+1$ given by

$$f(t)(k) = \begin{cases} \{t(\varnothing)\} & \text{if } k = n, \\ \{t(a) \mid a \subseteq n \text{ and } k \in a\} & \text{otherwise.} \end{cases}$$

Clearly, $\mathrm{ran}(f) \subseteq \mathrm{fin}(x)^{n+1}$ and, for all $t \in x^{\underline{\wp(n)}}$, t is the function on $\wp(n)$ given by

$$t(a) = \begin{cases} \bigcup f(t)(n) & \text{if } a = \varnothing, \\ \bigcup\left(\bigcap_{k \in a} f(t)(k) \setminus \bigcup_{k \in n \setminus a} f(t)(k)\right) & \text{otherwise.} \end{cases}$$

Hence, f is an injection from $x^{\underline{\wp(n)}}$ into $\mathrm{fin}(x)^{n+1}$.

Lemma

In $\mathcal{V}_{\mathrm{F}}$, $2^{\mathfrak{a}^{\underline{2^n}}} \nleq 2^{\mathrm{fin}(\mathfrak{a})^n}$, *where* $\mathfrak{a}$ *is the cardinality of the set of atoms.*

Proof. Let A be the set of atoms of $\mathcal{V}_{\mathrm{F}}$ and let $\mathfrak{a} = |A|$. Assume toward a contradiction that there exists an injection $f \in \mathcal{V}_{\mathrm{F}}$ from $\wp(A^{\underline{2^n}})$ into $\wp(\mathrm{fin}(A)^n)$. Let B be a finite support of f. Take an arbitrary $C \in [A \setminus B]^{2^n+1}$ and a $u \in C^{\underline{2^n}}$.

We say that a permutation π of A is *even* (*odd*) if π moves only elements of C and can be written as a product of an even (odd) number of transpositions. It is well known that a permutation of A cannot be both even and odd.

Basic Fraenkel Model

Now, let

$$\mathcal{E} = \{\pi(u) \mid \pi \text{ is an even permutation of } A\},$$

and let

$$\mathcal{O} = \{\sigma(u) \mid \sigma \text{ is an odd permutation of } A\}.$$

Clearly, $\{\mathcal{E}, \mathcal{O}\}$ is a partition of $C^{\underline{2^n}}$, $\pi(\mathcal{E}) = \mathcal{E}$ for all even permutations π, and $\sigma(\mathcal{E}) = \mathcal{O}$ for all odd permutations σ.

Basic Fraenkel Model

Now, let us consider $f(\mathcal{E})$. For each $t \in f(\mathcal{E})$, let $\sim_t$ be the equivalence relation on C such that, for all $a, b \in C$,

$$a \sim_t b \quad \text{if and only if} \quad \forall k < n\,(a \in t(k) \leftrightarrow b \in t(k)).$$

Clearly, for all even permutations π, $\pi(f(\mathcal{E})) = f(\mathcal{E})$.

For all odd permutations σ and all $t \in f(\mathcal{E})$, since $|C/{\sim_t}| \leqslant 2^n$ and $|C| = 2^n + 1$, there are $a, b \in C$ such that $a \neq b$ and $a \sim_t b$, and hence the transposition τ that swaps a and b fixes t, which implies that $\sigma(t) = (\sigma \circ \tau)(t) \in f(\mathcal{E})$ since $\sigma \circ \tau$ is even.

Hence, for all odd permutations σ of A, $\sigma(f(\mathcal{E})) = f(\mathcal{E})$, which implies that $f(\mathcal{O}) = f(\sigma(\mathcal{E})) = \sigma(f(\mathcal{E})) = f(\mathcal{E})$, contradicting the injectivity of f.

Main Theorems

Definition

Let x, y be arbitrary sets, let $\mathfrak{a} = |x|$, and let $\mathfrak{b} = |y|$.

- $x \preccurlyeq^* y$ means that $x = \emptyset$ or there is a surjection of y onto x, and $\mathfrak{a} \leqslant^* \mathfrak{b}$ means that $x \preccurlyeq^* y$.
- $\mathfrak{a} =^* \mathfrak{b}$ means that $\mathfrak{a} \leqslant^* \mathfrak{b}$ and $\mathfrak{b} \leqslant^* \mathfrak{a}$.

Fact

- $\mathfrak{a} \leqslant \mathfrak{b} \to \mathfrak{a} \leqslant^* \mathfrak{b} \to 2^{\mathfrak{a}} \leqslant 2^{\mathfrak{b}}$.
- $\mathfrak{a} = \mathfrak{b} \to \mathfrak{a} =^* \mathfrak{b} \to 2^{\mathfrak{a}} = 2^{\mathfrak{b}}$.

Main Theorems

Theorem

For all infinite cardinals $\mathfrak{a}$, $\mathrm{seq}^{1\text{-}1}(\mathfrak{a}) =^* \mathrm{fin}(\mathrm{fin}(\mathfrak{a})) =^* \mathrm{fin}^3(\mathfrak{a}) =^* \cdots =^* \mathrm{seq}(\mathfrak{a}) =^* \mathrm{seq}(\mathrm{fin}(\mathfrak{a}))$.

- $\mathrm{seq}^{1\text{-}1}(\mathfrak{a}) \leqslant \mathrm{fin}(\mathrm{fin}(\mathfrak{a}))$.
- $\mathrm{seq}(\mathrm{seq}(\mathfrak{a})) \leqslant \aleph_0 \, \mathrm{seq}(\mathfrak{a})$.
- $\mathrm{seq}(\mathfrak{a}) \leqslant \aleph_0 \, \mathrm{seq}^{1\text{-}1}(\mathfrak{a})$.
- $\aleph_0 \, \mathrm{seq}^{1\text{-}1}(\mathfrak{a}) \leqslant \mathrm{seq}(\mathfrak{a})$.
- $\aleph_0 \, \mathrm{seq}^{1\text{-}1}(\mathfrak{a}) \leqslant^* \mathrm{seq}^{1\text{-}1}(\mathfrak{a})$.

Corollary

$2^{\mathrm{seq}^{1\text{-}1}(\mathfrak{a})} = 2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))} = 2^{\mathrm{fin}^3(\mathfrak{a})} = \cdots = 2^{\mathrm{seq}(\mathfrak{a})} = 2^{\mathrm{seq}(\mathrm{fin}(\mathfrak{a}))}$.

Main Theorems

Theorem

For all infinite cardinals $\mathfrak{a}$ *and all* $n \in \omega$, $2^{(\mathrm{fin}(\mathfrak{a}))^n} = 2^{[\mathrm{fin}(\mathfrak{a})]^n}$.

Note that $(\mathrm{fin}(\mathfrak{a}))^n \not\leq^* [\mathrm{fin}(\mathfrak{a})]^n$ cannot be proved in ZF.

The proof of this theorem is a generalization of that of the following lemma:

Lemma (Läuchli 1961)

For all infinite cardinals $\mathfrak{a}$, $2^{\aleph_0 \cdot \mathrm{fin}(\mathfrak{a})} = 2^{\mathrm{fin}(\mathfrak{a})}$.

Proof of Läuchli's Lemma

Proof

Let A be a fixed set. For all n, k such that $n \leqslant k$, we define:

- $F_{n,k} : \wp([A]^n) \to \wp([A]^k)$ such that for all $X \subseteq [A]^n$,

$$F_{n,k}(X) = \{y \in [A]^k \mid \exists x \in X\,(x \subseteq y)\}$$

- $G_{n,k} : \wp([A]^n) \to \wp([A]^n)$ such that for all $X \subseteq [A]^n$,

$$G_{n,k}(X) = \{x \in [A]^n \mid \forall y \in [A]^k\,(x \subseteq y \to y \in F_{n,k}(X))\}$$

- For all $X \subseteq [A]^n$, $H_{n,k}(X) = G_{n,k}(X) \setminus X$.

Proof of Läuchli's Lemma

Fact

1. $X \subseteq Y \subseteq [A]^n \to F_{n,k}(X) \subseteq F_{n,k}(Y)$
2. $X \subseteq [A]^n \to X \subseteq G_{n,k}(X)$
3. $X \subseteq Y \subseteq [A]^n \to G_{n,k}(X) \subseteq G_{n,k}(Y)$
4. $X \subseteq [A]^n \to G_{n,k}(G_{n,k}(X)) = G_{n,k}(X)$
5. $X \subseteq [A]^n \to F_{n,k}(G_{n,k}(X)) = F_{n,k}(X)$
6. $F_{n,k} \upharpoonright \{X \subseteq [A]^n \mid G_{n,k}(X) = X\}$ is 1-1.
7. For all $X \subseteq [A]^n$ and all natural numbers m,
$$H^m_{n,k}(X) = G_{n,k}(H^m_{n,k}(X)) \setminus H^{m+1}_{n,k}(X)$$
8. $k \leqslant k' \wedge X \subseteq [A]^n \to G_{n,k}(X) \subseteq G_{n,k'}(X)$, and hence
$$\{X \subseteq [A]^n \mid G_{n,k'}(X) = X\} \subseteq \{X \subseteq [A]^n \mid G_{n,k}(X) = X\}$$

Proof of Läuchli's Lemma

Key Lemma

$X \subseteq [A]^n \to H_{n,k}^{n+1}(X) = \varnothing$

Corollary

$X \subseteq [A]^n \to H_{n,k}^{n}(X) = G_{n,k}(H_{n,k}^{n}(X))$

Proof of Läuchli's Lemma

Come back to the proof of Läuchli's Lemma

For all $X \subseteq \omega \times \mathrm{fin}(A)$ and all natural numbers i, n, m, we define:

$$X_{i,n}^{(0)} = X[\{i\}] \cap [A]^n$$
$$X_{i,n,m}^{(1)} = G_{n,2^i 3^n 5^n}(H^m_{n,2^i 3^n 5^n}(X_{i,n}^{(0)}))$$
$$X_{i,n,m}^{(2)} = F_{n,2^i 3^n 5^m}(X_{i,n,m}^{(1)})$$

Let

$$\Phi(X) = \bigcup_{i\in\omega} \bigcup_{n\in\omega} \bigcup_{m=0}^{n} X_{i,n,m}^{(2)}$$

Proof of Läuchli's Lemma

Note that if $m \leqslant n$, then

- $X^{(2)}_{i,n,m} = \Phi(X) \cap [A]^{2^i 3^n 5^m}$
- $$X^{(1)}_{i,n,m} = (F_{n,2^i 3^n 5^m} \upharpoonright \{Y \subseteq [A]^n \mid G_{n,2^i 3^n 5^m}(Y) = Y\})^{-1}(X^{(2)}_{i,n,m})$$
- $X^{(0)}_{i,n} = X^{(1)}_{i,n,0} \setminus (X^{(1)}_{i,n,1} \setminus (\cdots (X^{(1)}_{i,n,n-1} \setminus X^{(1)}_{i,n,n}) \cdots))$
- $X = \bigcup \{\{i\} \times X^{(0)}_{i,n} \mid i, n \in \omega\}$

Hence, Φ is an injection from $\wp(\omega \times \mathrm{fin}(A))$ into $\wp(\mathrm{fin}(A))$. $\square$

Proof of the Key Lemma

Key Lemma

$X \subseteq [A]^n \to H^{n+1}_{n,k}(X) = \varnothing$

Proof

Fix n, k with $n \leqslant k$. For convenience, we shall omit the subscripts. Consider the following two formulae:

- $\psi(X, x, y)$:

$$X \subseteq [A]^n \wedge x \in [A]^{\leqslant n} \wedge y \in \operatorname{fin}(A \setminus x) \wedge \forall z \in [y]^{n-|x|} (x \cup z \in X)$$

- $\phi(X, x)$:

$$X \subseteq [A]^n \wedge x \in [A]^{\leqslant n} \wedge \forall p \in \omega \, \exists y \in [A]^p \, \psi(X, x, y)$$

Proof of the Key Lemma

Claim

$X \subseteq [A]^n \wedge \phi(H(X), x) \rightarrow \exists u \subset x \ \phi(X, u)$

Assume that $X \subseteq [A]^n$ and $\phi(H(X), x)$. It suffices to prove that

$$\forall p \geqslant k \ \exists u \subset x \ \exists y \in [A]^p \ \psi(X, u, y) \qquad (\circledast)$$

Theorem (Ramsey)

There exists a function $R : \omega \times (\omega \setminus \{0\}) \times \omega \rightarrow \omega$ such that for all $n, r, p \in \omega$ with $r > 0$, whenever $S, Y_1, \ldots, Y_r$ are finite sets such that $[S]^n = \bigcup_{i=1}^r Y_i$ and $|S| \geqslant R(n, r, p)$, there is a $T \in [S]^p$ and an $i \in \{1, \ldots, r\}$ such that $[T]^n \subseteq Y_i$.

Proof of the Key Lemma

Let

$$p' = \max\{R(j, 2, p) \mid j \leqslant n\}$$

and let

$$p'' = R(k - |x|, 2^{|x|}, p')$$

Since $\phi(H(X), x)$, there is an $S \in [A]^{p''}$ such that $\psi(H(X), x, S)$. For each $u \subseteq x$, let

$$Y_u = \{w \in [S]^{k-|x|} \mid \exists v \in [w]^{n-|u|} \, (u \cup v \in X)\}$$

Claim

$$[S]^{k-|x|} = \bigcup_{u \in \wp(x)} Y_u$$

Proof of the Key Lemma

Then by Ramsey's Theorem, there is an $T \in [S]^{p'}$ and a $u \subseteq x$ such that

$$[T]^{k-|x|} \subseteq Y_u \tag{1}$$

Let

$$Z = \{v \in [T]^{n-|u|} \mid u \cup v \in X\}$$

and let $Z' = [T]^{n-|u|} \setminus Z$. Then $[T]^{n-|u|} = Z \cup Z'$, and since $|T| = p' \geqslant R(n - |u|, 2, p)$, it follows from Ramsey's Theorem that there is a $y \in [T]^p$ such that $[y]^{n-|u|} \subseteq Z$ or $[y]^{n-|u|} \subseteq Z'$. By (1), $[y]^{n-|u|} \cap Z \neq \emptyset$. Thus $[y]^{n-|u|} \subseteq Z$; that is, $\psi(X, u, y)$. If $u = x$, then $\psi(H(X), x, y)$ and $\psi(X, x, y)$, which is absurd. Hence ($\circledast$) is proved. $\square$

Proof of the Main Theorem

A direct n-dimensional generalization of Läuchli's lemma yields only

$$2^{\aleph_0 \cdot \operatorname{fin}(\mathfrak{a})^n} = 2^{\operatorname{fin}(\mathfrak{a})^n},$$

which is a trivial corollary of Läuchli's lemma.

But by generalizing the technique of encoding subsets of $[A]^n$, we can encode subsets of $[A]^{n_1} \times [A]^{n_m}$ by subsets of $[A]^{k_1} \times [A]^{k_m}$, where $n_i \leqslant k_i$ $(1 \leqslant i \leqslant m)$ and $k_1 < \cdots < k_m$, and each element of $[A]^{k_1} \times [A]^{k_m}$ is uniquely determined by its range. This gives a way to encode subsets of $\operatorname{fin}(A)^m$ by subsets of $[\operatorname{fin}(A)]^m$.

Some Open Problems

Question

Does ZF prove that $2^{2^{\mathrm{fin}(\mathfrak{a})}} = 2^{2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))}}$ for any infinite cardinal $\mathfrak{a}$?

Question (Läuchli 1961)

Does ZF prove that $2^{2^{\mathfrak{a}}} = 2^{2^{\mathfrak{a}+1}}$ for any infinite cardinal $\mathfrak{a}$?

Thank you