Square of Menger groups

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1. Classical Selection principle
2. Selection principle in groups
3. Main results

Foreword

Set-theoretic topology:

set theory—general topology—independent (ZFC) results.

Selection principle:

(1) selecting

(2) describe covering properties

(3) measure- and category-theoretic properties

(4) local properties in topological spaces

(5) function spaces

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Example 1

Definition (Borel 1919)

$X \subseteq \mathbb{R}$, strong measure zero :
$\forall(\varepsilon_n > 0)\exists \text{ interval } |I_n| < \varepsilon_n, X \subseteq \bigcup I_n.$

Conjecture (Borel conjecture)

Every strong measure zero set is countable.

Answer:
Sierpiński 1928: No, Under $\mathbf{CH}$.
Laver 1976: Yes, In Laver model.

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Definition (Rothberger 1938)

Topology space, Rothberger space:
$\forall$open cover $\mathcal{U}_n$, $\exists\, U_n \in \mathcal{U}_n$, $\{U_n\}$ is open cover.

Theorem (Fremlin-Miller 1988)

$X \subseteq \mathbb{R}$, strong measure zero $\iff$
$\forall$ compatible topology τ, (X, τ) is Rothberger.

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Example 2

Definition (Menger 1924)

Metric space, Menger basis property:
$\forall$basis $\mathcal{B}$, $\exists$ subcover $\{U_n\}$, $d(U_n) \to 0$.

Theorem (Hurewicz 1926)

Menger basis property $\iff$
$\forall$open cover $\mathcal{U}_n$, $\exists$ finite $U_n \subseteq \mathcal{U}_n$, $\{\bigcup U_n\}$ *open cover.*

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Definition (Franklin 1965)

Topology space X, Fréchet-Urysohn space :
$\forall A \subseteq X$, $\overline{A} = [A]_{\mathsf{seq}} = \{x \in X : \exists \{a_n\} \to x, a_n \in A\}$.

Tychonoff space X,
$C(X)$: continuous functions $f\colon X \to \mathbb{R}$,
pointwise convergence topology.

Question (Gerlits-Nagy 1982)

When $C(X)$ is Fréchet-Urysohn space?

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Definition (Gerlits-Nagy, 1982)

(1) Open cover $\mathcal{U}$, Ω-cover: $\forall$ finite $F \subseteq X$, $\exists U \in \mathcal{U}$, $F \subseteq U$.

(2) Open cover $\mathcal{U}$, Γ-cover: $\forall x \in X$, $\{U \in \mathcal{U} : x \notin U\}$ finite.

(3) Topology space X, γ-space: $\forall\Omega$-cover $\mathcal{U}$ $\exists\Gamma$-cover $\mathcal{V} \subseteq U$.

Theorem (Gerlits-Nagy 1982)

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Modern definition

Definition (Scheepers 1996)

Let $\mathcal{A}$ and $\mathcal{B}$.

(1) $\mathsf{S}_1(\mathcal{A},\mathcal{B})$: $\forall\langle\mathcal{U}_n\rangle\in\mathcal{A}^{\mathbb{N}}\ \exists U_n\in\mathcal{U}_n$, $\{U_n\}\in\mathcal{B}$.

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Explained by picture

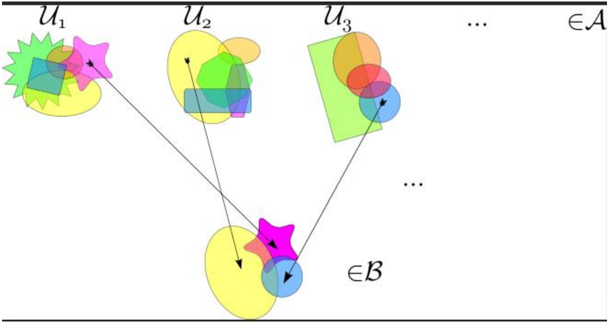

Figure: $\mathsf{S}_1(\mathcal{A},\mathcal{B})$

Scheepers Diagram

$\Pi(\mathcal{A},\mathcal{B})$, $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\mathsf{fin}}, \mathsf{U}_{\mathsf{fin}}, \binom{\ }{\ }\}$, $\mathcal{A},\mathcal{B} \in \{\mathrm{O}, \Gamma, \Omega\}$.

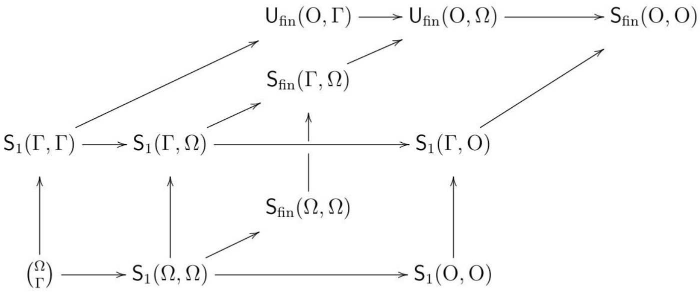

Figure: Scheepers Diagram

Product

Theorem (Todorćević 1995)

$\exists X, Y \in \mathrm{S}_1(\Omega, \Gamma)$, $X \times Y$ *is not Lindelöf.*

Theorem (Miller-Tsaban-Zdomskyy 2013)

$\mathbf{CH}$, $\exists X, Y \subseteq \mathbb{R}, \mathrm{S}_1(\Omega, \Gamma)$, $X \times Y \notin \mathrm{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$.

Theorem (Zdomskyy 2018)

In Miller model: $\forall X, Y \subseteq \mathbb{R}, \mathrm{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O}) \implies X \times Y \in \mathrm{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$.

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Groups

Definition (M. Tkačenko 1998)

Topological group G, o-bounded:
$\forall U_n \ni e$, $\exists$ finite F_n, $G = \bigcup_{n\in\omega} F_n * U_n$.

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$(G, *, e, \tau)$. $\mathcal{N}_e$: all open neighborhoods of e.
$U \in \mathcal{N}_e$, $g \in G$, let $g * U := \{g * u : u \in U\}$.
$\mathrm{O}_{\mathrm{nbd}}$: covers of the form $\{g * U : g \in G\}$, for $U \in \mathcal{N}_e$.

Definition

Assume that $(G, *, e, \tau)$ is a topological group.
G is Menger-bounded: $\mathrm{S}_{\mathrm{fin}}(\mathrm{O}_{\mathrm{nbd}}, \mathrm{O})$.

Fact: $\mathbf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O}) \Longrightarrow \mathrm{S}_{\mathrm{fin}}(\mathrm{O}_{\mathrm{nbd}}, \mathrm{O})$.

$\frac{1}{2}$-answer

Theorem (Krawczyk-Michalewski 2003)

CH. $\exists$ *Menger-bounded group* $G, H \leq \mathbb{R}^\omega$, $G \times H$ *is not Menger-bounded.*

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CH. $\exists$ *Menger-bounded group* $G \leq \mathbb{Z}^\omega$, G^2 *is not Menger-bounded.*

Theorem (Mildenberger 2008)

$\mathfrak{r} \geq \mathfrak{d}$. $\forall k$, $\exists$ $G \leq \mathbb{Z}^\omega$,G^k *is Menger-bounded,* G^{k+1} *is not Menger-bounded.*

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Remain open

Problem (M. Tkačenko)

Is it consistent that for each Menger-bounded G, H, $G \times H$ *Menger-bounded?*

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Our motivation

Question (P. Szewczak, B. Tsaban and L. Zdomskky 2018)

Is there, consistently, a Menger topological group whose square is not Menger?

Main results

Theorem (He, Peng and Wu 2020)

(1) $cov(\mathcal{M}) = \mathfrak{c}$. *For each* $n \geq 1$, *there is a subgroup of* $\mathbb{Z}^{\mathbb{N}}$ *such that* G^n *is Menger but* G^{n+1} *is not Menger-bounded.*

(2) *For each* $n \geq 1$, *there is a subgroup of* $\mathbb{R}^{\omega_1}$ *such that* G^n *is Menger but* G^{n+1} *is not Lindelöf.*

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Corollary

1. *Menger group square problem is independent with ZFC in the metrizable sense.*
2. *Menger group square problem is negative in nonmetrizable sense.*

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Preliminary: Cardinal invariants

- For any $f, g \in \mathbb{N}^{\mathbb{N}}$, $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n.
- $\mathcal{D} \subseteq \mathbb{N}^{\mathbb{N}}$ is a dominating family if for each $f \in \mathbb{N}^{\mathbb{N}}$ there exists $g \in \mathcal{D}$ such that $f \leq^* g$. $\mathfrak{d}$ is the least cardinality among all dominating families.
- $cov(\mathcal{M})$ is the least cardinality among all families of comeager subsets of $\mathbb{N}^{\mathbb{N}}$ which has empty intersection.

Theorem (Hurewicz)

A set of reals X has Menger's property if, and only if, no continuous image of X in $\mathbb{N}^{\mathbb{N}}$ is dominating.

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Preliminary: Walks on ordinal

Definition

- $[\omega_1]^2$ is the set of all subset of ω_1 of size 2.
- $osc \colon [\omega_1]^2 \to \mathbb{N}$ is a function. Denote $osc_\alpha \colon \alpha \to \mathbb{N}$ by $osc_\alpha(\xi) = osc(\{\alpha, \xi\})$ for $\xi < \alpha < \omega_1$.
- $T = \{osc_\alpha | \beta : \beta \leq \alpha\}$ and $level_T(\beta) = \{ocs_\alpha | \beta : \beta \leq \alpha < \omega_1\}$ for any $\beta < \omega_1$.

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- $\overline{\mathbb{N}}$ is the one-point compactification $\mathbb{N} \cup \{\infty\}$ of $\mathbb{N}$ with discrete topology.
- $\overline{\mathbb{N}}^{\uparrow\mathbb{N}}$ is the set of all increasing functions $f : \mathbb{N} \to \overline{\mathbb{N}}$.
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Preliminary: Linear algebra

Definition

- View $\mathbb{R}$ as a vector space over $\mathbb{Q}$. $span_{\mathbb{Q}}(X)$ is the vector subspace generated by X for any $X \subseteq \mathbb{R}$.
- For any $\{\theta_\alpha : \alpha < \omega_1\} \subseteq \mathbb{R}$ and $x \in span_{\mathbb{Q}}(\{\theta_\alpha : \alpha < \omega_1\})$, the height of x in $\{\theta_\alpha : \alpha < \omega_1\}$ is the least α such that $x \in span_{\mathbb{Q}}(\{\theta_\xi : \xi \leq \alpha\})$.
- A collection of real numbers is rationally independent if none of them can be written as a linear combination of the other numbers in the collection with rational coefficients.
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Construction of subgroup of $\mathbb{R}^{\omega_1}$

Recall Peng-Wu's construction of a group G such that G^n is Lindelöf but G^{n+1} is not Lindelöf.
Fix n.

- Construct $h\colon [0,1) \to \mathbb{Q}$ and a sequences of comeager set $\{X_m \subseteq \mathbb{R}^m : n \leq m \in \mathbb{N}\}$.
- Choose $\{\theta_\alpha : \alpha < \omega_1\}$ such that any $m \geq n$ and sequence $x_i \in span_{\mathbb{Q}}(\{\theta_\alpha : \alpha < \omega_1\})$, $i < m$ of elements with increasing heights, $(x_0, x_1, \cdots, x_{m-1}) \in X_m$.
- Define $\{w_\beta : \beta < \omega_1\} \subseteq \mathbb{R}^{\omega_1}$ as follows:

$$w_\beta(\xi) = \begin{cases} h(frac(\theta_\xi osc(\xi,\beta) + \theta_\beta) & : \xi < \beta \\ 0 & : \xi \geq \beta \end{cases}$$

and G be the subgroup of $(\mathbb{R}^{\omega_1}, +)$ which is generated by $\{w_\beta : \beta < \omega_1\}$.

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When the group is Lindelöf ?

In fact, h is continuous on a comeager set D.

Definition

Let $Y = \{\theta_\alpha : \alpha < \omega_1\} \subseteq \mathbb{R}$. We call Y reads $(h, \vec{X})$ if

(1) Y is a set of irrational real numbers.

(2) $Y \cup \{1\}$ is rationally independent.

(3) For any $p \in \mathbb{N}, \beta < \alpha$, $frac(p\theta_\beta + \theta_\alpha) \in D$.

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In fact, h is continuous on a comeager set D.

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Let $Y = \{\theta_\alpha : \alpha < \omega_1\} \subseteq \mathbb{R}$. We call Y reads $(h, \vec{X})$ if

(1) Y is a set of irrational real numbers.

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Definition

Let $T = \{osc_\gamma|\beta : \beta < \gamma < \omega_1\}$. For each $\beta < \omega_1$, define the partition $\mathcal{P}_\beta$ of $[\beta, \omega_1)$ by $\{\{\gamma \geq \beta : osc_\gamma|\beta = t\} : t \in level_T(\beta)\}$. We call $Y = \{\theta_\alpha : \alpha < \omega_1\}$ is solid Menger if for any $\beta < \omega_1$, the product of any finite many spaces from $\mathcal{P}_\beta(Y) = \{\{\theta_\gamma : \gamma \in P\} : P \in \mathcal{P}_\beta\}$ is Menger.

Theorem

Assume Y reads $(h, \vec{X})$. If Y is solid Menger, then G^n is Menger but G^{n+1} is not Menger.

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There exists a solid Menger space Y which reads $(h, \vec{X})$ in ZFC.

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Tools for constructing Menger space

Definition

Let $1 \leq n \in \mathbb{N}$ and $\{X_i \in [\overline{\mathbb{N}}^{\uparrow\mathbb{N}}]^{\mathfrak{d}} : i < n\}$. We call $\prod_{i<n} X_i$ is n-$\mathfrak{d}$-unbounded if for each $g \in \mathbb{N}^{\mathbb{N}}$, there are $A_i \in [X_i]^{<\mathfrak{d}}$ such that for any $\vec{x} \in \prod_{i<n} (X_i \setminus A_i)$, $\min(\vec{x}) \not\leq^* g$.

Lemma ($cf(\mathfrak{d}) = \mathfrak{d}$)

Let $1 \leq n \in \mathbb{N}$, and $\{X_i : i < n\}$ be a sequence of subsets of $\overline{\mathbb{N}}^{\uparrow\mathbb{N}}$ with size $\mathfrak{d}$ and containing $\mathbb{Q}_\infty$. If $\prod_{i<n} X_i$ is n-$\mathfrak{d}$-unbounded, then $\prod_{i<n} X_i$ is Menger.

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Construction of subgroup of $\mathbb{R}$

$n = 2$, subgroup of $\mathbb{R}^\omega$:

Theorem ($cov(\mathcal{M}) = \mathfrak{d}$)

There exists a Menger subgroup G of $\mathbb{R}^\omega$ such that G^2 is not Menger.

Sketch of proof.

$P = \{p \in \mathbb{R}^\omega : \forall k < \omega (p(k) \in \mathbb{Q})\ \&\ \exists l < \omega \forall k \geq l (p(k) = 0)\}$
$= \{p_k : k < \omega\}$.

P_0 : a maximal linear independent subset of P.

$\mathcal{D} = \{f_\alpha : \alpha < \mathfrak{d}\}$: a dominating family and also closed under finite modification.

For each $\alpha < \mathfrak{d}$, define

$$W_{f_\alpha} = \bigcap_{g =^* f_\alpha} \bigcup_{k<\omega} \mathcal{B}_\rho(p_k, \frac{1}{g(k)+1}).$$

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Sketch of proof (continue)

Proof.

Construct $\{ x_\alpha, y_\alpha \in \mathbb{R}^\omega : \alpha < \mathfrak{d}\}$ by transfinite recursion.
H_α: the group generated by $\{x_\beta, y_\beta : \beta < \alpha\}$
G_α: the group generated by $\{x_\beta, y_\beta : \beta \leq \alpha\}$.
$G = \bigcup_{\alpha<\mathfrak{d}} G_\alpha$.
By induction, we will make sure that the following requirements are satisfied.

(1) For any $\alpha < \mathfrak{d}$, $P_0 \cup \{x_\beta, y_\beta : \beta \leq \alpha\}$ is linear independent.
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$n \geq 3$, subgroup of $\mathbb{R}$:

Lemma

Let $C \in \mathbb{Z}^{m\times(n+1)}$ with $rank(C) = m \leq n$, $\vec{q} \in \mathbb{R}^m$ and $f \in \mathbb{N}^{\mathbb{N}}$. Then there exists $\vec{y} \in (\mathbb{R}/\mathbb{Q})^{n+1}$ such that $\vec{y} \cdot C^T = \vec{q}$ and $\sum_{j\leq n} |h(y_j)(k)| > f(k)$ for all $k \in \mathbb{N}$.

Lemma

Let $f \in \mathbb{N}^{\mathbb{N}}$, $Y = \{\vec{y} \in (\mathbb{R}/\mathbb{Q})^{n+1} : \forall k \in \mathbb{N} \, (\sum_{j\leq n} |h(y_j)(k)| > f(k))\}$, $C \in \mathbb{Z}^{m\times(n+1)}$ with $rank(C) = m \leq n$ and W be comeager subset of $\mathbb{R}^m$. Then $\{\vec{y} \in Y : \vec{y} \cdot C^T \in W\}$ is comeager set in Y.

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A new conception: W-rich

Definition

Let $W \subseteq \mathbb{R}^n$ be comeager set, $A \in \mathbb{Q}^{n^2}$ be invertible matrix, $\vec{x} \in \mathbb{R}^m$ and $1 \leq m \leq n-1$.

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$W \subseteq \mathbb{R}^n$: comeager set, $\vec{r} : r_0 = 0 < r_1 < r_2 < \cdots < r_k < n-1$, $\vec{A} = \{A_j \in \mathbb{Q}^{(n-r_j)^2} : j < k\}$: invertible matrixes and $\vec{x} \in \mathbb{R}^n$.

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Properties of W-rich

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$C \in \mathbb{Z}^{n \times (n+1)}$, $\vec{z} \in \mathbb{R}^n$, *and* W *be comeager in* $\mathbb{R}^n$. *If* $\vec{z}$ *is* W*-rich, then* $\{\vec{y} \in Y : \vec{z} + \vec{y} \cdot C^T$ *is* W*-rich*$\}$ *is comeager in* Y.

Lemma

Let $W \subseteq \mathbb{R}^n$ *be comeager. If* $\vec{0} \in W$ *and* $W_{\vec{0}|n \setminus r}$ *is comeager in* $\mathbb{R}^r$ *for any* $1 \leq r < n$. *Then* $\vec{0}$ *is* W*-rich.*

Properties of W-rich

Lemma

Let $f \in \mathbb{N}^{\mathbb{N}}$,
$Y = \{\vec{y} \in (\mathbb{R}/\mathbb{Q})^{n+1} : \forall k \in \mathbb{N}\, (\sum_{j \leq n} |h(y_j)(k)| > f(k))\}$,
$C \in \mathbb{Z}^{n\times(n+1)}$, $\vec{z} \in \mathbb{R}^n$, *and* W *be comeager in* $\mathbb{R}^n$. *If* $\vec{z}$ *is* W*-rich, then* $\{\vec{y} \in Y : \vec{z} + \vec{y} \cdot C^T \text{ is } W\text{-rich}\}$ *is comeager in* Y.

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Construction of subgroup of $\mathbb{R}$

Theorem ($cov(\mathcal{M}) = \mathfrak{d} = cf(\mathfrak{d})$)

For any $n \geq 1$, There is a subgroup G of $\mathbb{R}$ such that G^n is Menger but G^{n+1} is not Menger.

Sketch of proof.

$\mathbb{R}/\mathbb{Q}$: $\{0\} \cup (\mathbb{R} \setminus \mathbb{Q})$

$h \colon \mathbb{R}/\mathbb{Q} \to \overline{\mathbb{N}}^{\uparrow\mathbb{N}}$: a fixed homeomorphism embedding with $h(\mathbb{Q}[\sqrt{2}]/\mathbb{Q}) = \mathbb{Q}_\infty$.

P_0 : a maximal linear independent subset of $\mathbb{Q}[\sqrt{2}]/\mathbb{Q}$.

$\mathcal{D} = \{f_\alpha : \alpha < \mathfrak{d}\}$: a dominating family and also closed under finite modification.

For each $\alpha < \mathfrak{d}$, define

$$W_{f_\alpha} = \{\vec{x} \in (\mathbb{R}/\mathbb{Q})^n : \exists^\infty p \in \mathbb{N}\, (f_\alpha(p) < \min\{h(x_i)(p) : i < n\})\}.$$

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Sketch of proof (continue)

Proof.

Construct $\{\vec{x}_\alpha \in \mathbb{R}^{n+1} : \alpha < \mathfrak{d}\}$ by transfinite recursion.
H_α: the group generated by $\{x_{\beta,j} : \beta < \alpha, j \leq n\}$.
G_α: the group generated by $\{x_{\beta,j} : \beta \leq \alpha, j \leq n\}$.
$G = \bigcup_{\alpha<\mathfrak{d}} G_\alpha$.
By induction, we will make sure that the following requirements are satisfied.

- $G_\alpha \subseteq \mathbb{R}/\mathbb{Q}$.
- For any $\alpha < \mathfrak{d}$, $P_0 \cup \{x_{\beta,j} : \beta \leq \alpha, j \leq n\}$ is linearly independent.
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Thanks for your attention!