A priority argument in $0^{(4)}$

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joint work with
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R.e. degrees and Priority Methods

- A set A is *recursively enumerable (r.e.)* if $A = \operatorname{dom} f$ for some partial recursive function f.
- The halting problem $K = \{e \mid \Phi_e(e) \downarrow\}$ is an example of non-recursive r.e. set.
- 1944 Post's Problem: Is there a non-recursive incomplete r.e. degrees?
- 1957,1956 Friedberg-Muchnik Theorem: Yes (By a priority argument (with injuries!)).

Priority Methods (Tree methods)

Three steps:

- Write down a list of requirements.
- Arrange them on a priority tree, to help deciding which requirement has higher priority than the others.
- Resolve the conflicts of each requirements to get every requirement satisfied.

Complexity of priority methods

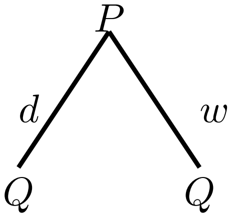
How many jumps of $\varnothing$ do we need to locate the final outcome of each requirement of the final set.
Examples in r.e. degrees.

- $0'$. Friedberg-Muchnick.
  There are A, B such that $A|_T B$.
- $0''$. Minimal pair.
  There are A, B such that $A|_T B$ and they bounds only recursive sets.
- $0'''$. Slaman Triple.
  There are A, B, C such that A, B form a minimal pair and with the help of any nonrecursive set $X \leq_T A$, $B \oplus X$ computes C, where we require $C \not\leq_T B$) to avoid triviality.
  Another strengthen of the Slaman Triple would require C to compute A, which will be called *strong minimal pair*, yet another strengthening would be a *double sided minimal pair*
- $0''''$. There is no strong minimal pair.

Friedberg-Muchnik $0'$

There are A, B such that $A|_T B$.
Requirements are

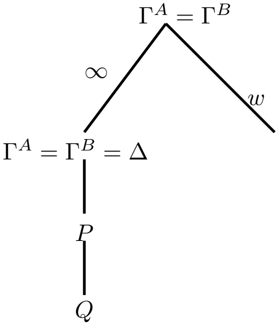
$$P(\Psi) : A \neq \Psi^B$$
$$Q(\Phi) : B \neq \Phi^A$$



Minimal Pair $0''$

There are A, B such that $A|_T B$ and they bounds only recursive sets.
Requirements are

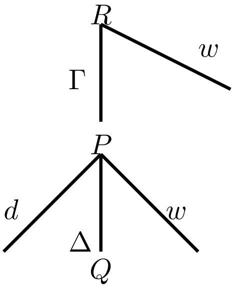
$$P(\Psi) : A \neq \Psi$$
$$Q(\Phi) : B \neq \Phi$$
$$N(\Gamma) : \Gamma^A = \Gamma^B \rightarrow \Gamma^A = \Gamma^B = \Delta$$



Slaman Triple $0'''$

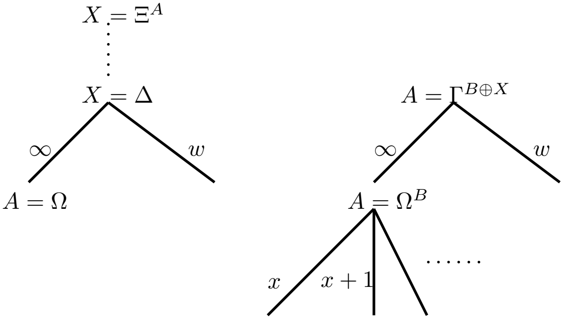
There are A, B, C such that A, B form a minimal pair and with the help of any nonrecursive set $X \leq_T A$, $B \oplus X$ computes C, where we require $C \not\leq_T B$) to avoid triviality.
Requirements are

$$R : U = \Phi^A \to C = \Gamma^{B \oplus U} \text{ or } U = \Delta$$
$$P : C \neq \Phi^B$$
$$Q : A \neq \Psi$$



A warm up

Suppose $A \not\leq_T B$



For Ω to succeed, the following condition must hold at any stage

$$\Omega(x) \downarrow \to \omega(x) > \gamma(x)$$

That is, while $\Omega(x) \downarrow$, it puts a restraint on X.

$0''''$

Theorem

There is no strong minimal pair.

Given $A \not\leq_T B$, we would like to exhibit an X such that the following requirements $\mathcal{R}(X)$ are satisfied:

$$P(X) : X = \Xi^A$$
$$G_e(X) : A \neq \Gamma_e^{B \oplus X}$$
$$D_e(X) : X \neq \Delta_e$$

Suppose we are to construct a set U so that $\mathcal{R}(U)$ are satisfied. Usually for a requirement like $G_e(U)$, if we have infinitely many expansion of $A = \Gamma_e^{B \oplus U}$, we are able to detect a divergent point of it using $0''$. However, suppose we fail to detect a divergent point, the outcome of the failure would require $0'''$ to identify. Moreover, below this outcome, we should work on another set V instead of U.

This gives some hint to the need of $0''''$.

Uniform and Nonuniform (1)

Our problem can be phrased as

$$Q(i) \to P(f(i), i)$$

Where

- i and $f(i)$ are the indices of r.e. sets
- Q and P are relations that are invariant of r.e. degrees. That is, if $W_i \equiv_T W_j$, then $Q(i) \leftrightarrow Q(j)$.
- f is a function that gives us a solution of the problem.

Now we can consider the complexity of this f.

- In a uniform construction, this f is always recursive.
- That is, if we can show this f is not recursive, then we know that our construction can not be uniform.

Uniform and Nonuniform (2)

- An f is n-d.n.c. if $\forall x, f(x) \neq \Phi_x^{0^{(n-1)}}(x)$.
- An f is n-f.p.f. if

  Case $n = 1$, $\forall x W_{f(x)} \neq W_x$.
  Case $n = 2$, $\forall x W_{f(x)} \neq^* W_x$.
  Case $n = 3$, $\forall x W_{f(x)} \not\equiv_T W_x$.
  Case $n \geq 3$, $\forall x W_{f(x)}^{(n-3)} \not\equiv_T W_x^{(n-3)}$.
- (Kucera) Each n-d.n.c. function computes an n-f.p.f. function. And vice versa.
- $0^{(n-1)}$ does not compute any n-d.n.c. function.
- Thus if $f \leq_T 0^{(n-1)}$, then f is not an n-f.p.f. function.
- In particular, if $f \leq_T 0''$, then there exists e such that $W_{f(e)} \equiv_T W_e$.

Uniform and Nonuniform (3)

Proposition

Recall $Q(i) \to P(f(i), i)$. Suppose we have a recursive function g such that for any j, we have $Q(g(j))$ and $\neg P(j, g(j))$, then $f \not\leq_T 0''$.

Proof.

Suppose $f \leq_T 0''$, then $f \circ g \leq_T 0''$, hence there is some j such that $W_{f(g(j))} \equiv_T W_j$. Now, $Q(g(j))$ gives $P(f(g(j)), g(j))$, which is $P(j, g(j))$ contradicting that $\neg P(j, g(j))$. □

Lemma

Given any set U, we can uniformly find sets A, B such that

- $A \not\leq_T B$
- $U \leq_T A \to A \leq_T B \oplus U$ *or* $U \leq_T \varnothing$.

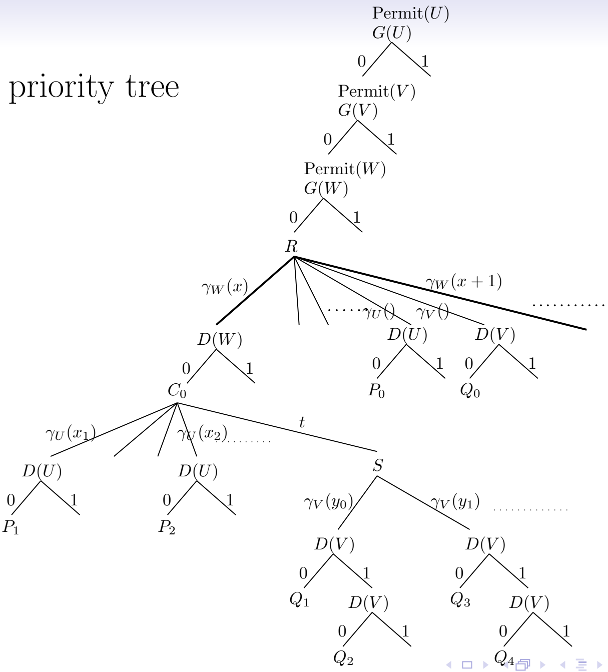
Corollary

The construction for nonexistence of strong minimal pair cannot be uniform.

Requirements are the following:

$$R_\Psi : A \neq \Psi^B$$
$$N_\Phi : U = \Phi^A \to A = \Gamma^{B \oplus U} \text{ or } U = \Delta$$

priority tree



Thank you!