Higher Randomness Theory and Its Applications

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History

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1. Incompressible;
2. No distinguish property;
3. Unpredictable.

Notations

We always identify a reals as its binary expansion.

Classical Randomness

People interested in classical randomness live in a computable world. To them, randomness means random relative to the computable world.

Kolmogorov complexity

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If U is a universal Turing machines, both C_U and K_U have the minimality property.

Martin-Löf test

Definition (Martin-Löf)

(i) Given a real x, a Σ^0_1 Martin-Löf test is a computable collection $\{V_n : n \in \mathbb{N}\}$ of x-c.e. sets such that $\mu(V_n) \leq 2^{-n}$.

(ii) Given a real x, a real y is said to pass the $\Sigma^0_1(x)$ Martin-Löf test if $y \notin \bigcap_{n\in\omega} V_n$.

(iii) Given a real x, a real y is said to be 1-x-random if it passes all $\Sigma^0_1(x)$ Martin-Löf tests.

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(iii) Given a real x, a real y is said to be 1-x-random if it passes all $\Sigma^0_1(x)$ Martin-Löf tests.

There exists a universal c.e. Martin-Löf test.

Betting strategy

Definition

1. A martingale is a function $f : 2^{<\omega} \mapsto \mathbb{R}$ such that for all $\sigma \in 2^{<\omega}$, $f(\sigma) = \frac{f(\sigma^\frown 0)+f(\sigma^\frown 1)}{2}$.
2. A martingale f is said to succeed on a real y if $\limsup_n f(y \upharpoonright n) = \infty$.

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Theorem (Schnorr)

(1) does not exist and the others are equivalent.

Chaitin's Ω

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Theorem (Chaitin)

Ω_U is a random real.

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How to compare randomness? Does higher complexity mean higher randomness?
Obviously a random real can compress itself.
But can it do more? Does richer mean more power?
It turns out that more random means less power.

Higher Randomness Theory

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But, really...?

Constructibility

We may perform recursive operators over sets just like numbers.

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The constructible universe L is a convenient world to perform recursive operators.

Gödel's Theorem

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L satisfies ZFC.

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So it is safe to say that mathematicians live in a "computable world".

The real world for recursion theorists

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$L_{\omega_1^{\mathrm{CK}}}$ is the world for recursion theorists.

Computation in $L_{\omega_1^{\mathrm{CK}}}$

For logicians, a computation is a Σ_1-procedure. So a computation in $L_{\omega_1^{\mathrm{CK}}}$ is a searching procedure over the elements of $L_{\omega_1^{\mathrm{CK}}}$.

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The time in $L_{\omega_1^{\mathrm{CK}}}$ is longer than the space. This results in that some techniques in computable world do not work in $L_{\omega_1^{\mathrm{CK}}}$.

Spector-Gandy's Theorem

Theorem (Spector, Gandy)

If set $A \subseteq \omega$ is Π^1_1 if and only if there is a Σ_1 formula φ so that for any number n,

$$n \in A \leftrightarrow L_{\omega_1^{\mathrm{CK}}} \models \varphi(n).$$

Π^1_1-randomness

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Definition

A real x is Π^1_1-random if it does not belong to any Π^1_1-null set.

Some earlier results.

Theorem (Sacks)

Every Δ^1_1-random real is Σ^1_1-random.

Theorem (Martin-Löf)

The set of Δ^1_1-random reals is Σ^1_1.

Recent results (1)

Theorem (Hjorth and Nies)

- *There is a universal Π^1_1-ML-test.*
- *Kucera-Gacs result hold for Π^1_1-ML-randomness*

Theorem (Chong, Nies, Y; Chong, Y)

- *A Δ^1_1-random real x is Π^1_1-random iff $\omega_1^x = \omega_1^{\mathrm{CK}}$ (also due to Stern);*
- *Strongly Π^1_1-ML-random is different than Π^1_1-ML-random;*
- *x is never continuous Π^1_1-random iff $x \in L_{\omega_1^x}$.*
- *If x is Π^1_1-random and $y \leq_h x$ is not hyperarithmetic, then y is hyperarithmetic equivalent to a Π^1_1-random real.*

Recent results (2)

Theorem (Kjos-Hanssen, Nies, Stephan, Y.)

- *A Δ^1_1-Kurtz-random real x is Π^1_1-random iff x is Δ^1_1-dominated;*
- *For any real z, there is a z-Kurtz random real $x \geq_T z$.*

Theorem (Greenberg and Monin; Greenberg, Laurent and Monin)

- *Low for Π^1_1-random implies hyperarithmetic.*
- *Strongly Π^1_1-ML random is different than Π^1_1-random.*

An application to hyperarithmetic degree theory

Theorem (Kripke)

If A is a null set of downward closed hyperdegrees without $\mathbf{0}$, then the upper-ward closure of A is null.

Proof.

Applying higher Demuth theorem by Chong and Y. □

Mauldin's Theorem

Theorem (Mauldin)

There is a Π^1_1 set A so that there are no Borel set B and F_σ null set X for which $B \subseteq A \subseteq B \cup X$.

Proof.

Let $A = \{x \oplus y \mid x \in L_{\omega_1^{x \oplus y}}[y]\}$. Applying Kurtz-randomness theory. □

The result can be generalized to σ-ideals.

Banach's theorem

A function $f\colon \mathbb{R} \to \mathbb{R}$ has Luzin-(N)-property if it maps a null set to a null set.

Theorem (Banach)

If f is continuous and has Luzin-(N)-property, then for almost every point y, $f^{-1}(y)$ is countable.

Proof.

Luzin-(N)-property says that f does not increase randomness. But randomness has no power. □

The result can be generalized to any measurable function.

We believe that this is just *a tip of iceberg*.

Further readings

An introduction to Kolmogorov complexity, Li and Vitany, 2008.

Computability and randomness, Nies, 2012.

Algorithmic randomness and complexity, Downey and Hirschfeldt, 2010.

Recursion theory, Chong and Yu, 2015.

Thanks!