Identity crisis for measurable and strongly compact cardinals

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Gödel's cerebrated incompleteness theorem shows the incompleteness of ZFC. One direction of modern set theory is to seek for an ultimate axiomatized system of set theory extending ZFC. Large cardinal axioms are nice candidates to be included in such extensions.

Besides some internal and external justification of large cardinals axioms, one naïve reason to accept them is simply because large cardinal essentially forms a strictly increasing hierarchy.

Our talk focus on some structural properties of three basic large cardinal notions.

Measurable cardinal

Definition (Ulam, 1930s)

A measurable cardinal κ is an uncountable cardinal κ such that there exists a κ-additive, non-trivial, 0-1-valued measure on the power set of κ.

Equivalently, κ is a measurable cardinal if and only if it is an uncountable cardinal with a κ-complete, non-principal ultrafilter.

Measurable cardinal is the first large cardinal notion originated from a pure mathematical problem.

Strongly compact cardinal

Recall that $\mathcal{L}_{\lambda\eta}$ is the infinitary predicate logic allowing $< \lambda$ many connectives and $< \eta$ many quantifiers in any formulae.

Definition (Tarski, 1962)

A strongly compact κ is an uncountable cardinal such that for any collection Γ of $\mathcal{L}_{\kappa\kappa}$ sentences, if Γ is κ-satisfiable, then Γ is satisfiable.

Equivalently, κ is a strongly compact cardinal if and only if for any set S, any κ-complete filter on S can be extended to a κ-complete ultrafilter on S.

Supercompact cardinal

Definition (Solovay-Reinhardt)

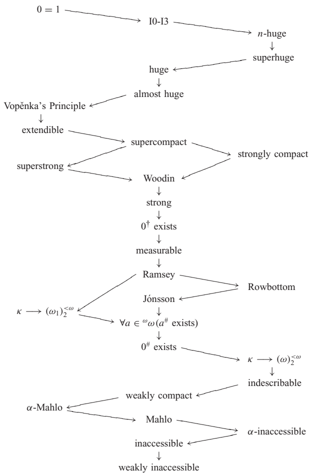
A supercompact cardinal is an uncountable cardinal κ if for every A such that $|A| \geq \kappa$ there exists a normal measure over $P_\kappa(A)$

Supercompact is originally defined as a normalized version of strongly compact cardinal.

Chart of Large Cardinals

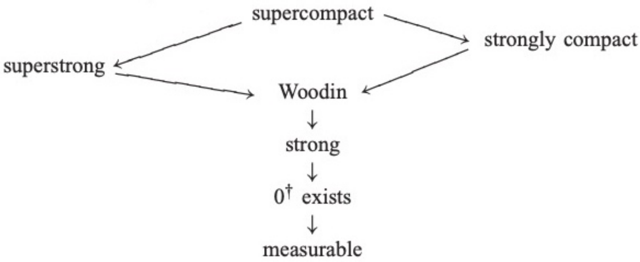
Chart of Cardinals

The arrows indicates direct implications or relative consistency implications, often both.



From Kanamori's "The higher infinite"

a bit closer..


supercompact
strongly compact
superstrong
Woodin
strong
$0^\dagger$ exists
measurable

Elementary embeddings and large cardinals

We will also use the elementary embedding to character large cardinals.

Theorem

For any regular κ,

1. *κ is measurable iff there is $j : V \to M \subseteq V$ with critical point κ.(Keisler)*
2. *κ is strongly compact iff for any $A \subset Ord$ there is $j : V \to M \subseteq V$ and $B \in M$ with critical point κ, $|B| < j(\kappa)$ and $A \subseteq B$.(Keisler-Tarski)*
3. *κ is supercompact iff for any $A \subset Ord$ there is $j : V \to M \subseteq V$ with critical point κ and $j[A] \in M$.(Solovay-Reinhardt)*

Corollary

supercompact $\to$ strongly compact $\to$ measurable.

One prominent problem in 1960s was to study whether these implications are revertible. Consistently, both of them are not.

measurable $\not\to$ strongly compact

Fact (Solovay)

Suppose κ is measurable cardinal and U is a measure on κ. Let $\bar{U} = U \cap L[U]$. Then $L[U] \models V = L[\bar{U}]$ and $\bar{U}$ is a measure on κ.

Theorem (Vopenka-Hrbacek)

If there is a strongly compact cardinal, then $V \neq L(A)$ for any set A.

Later development of inner model theory indicates that strongly compact cardinal implies $Con(ZFC + \exists$ measurable cardinal).

strongly compact $\not\to$ supercompact

Theorem (Menas)

1. *A measurable limit of strongly compact is strongly compact*
2. *The least cardinal satisfying (1) is not supercompact*

identity crisis for strongly compact cardinal

On the other hand, Magidor shows that it is consistent to have both arrows revertible.

Theorem (Magidor)

1. *If κ is supercompact, then there is a forcing extension in which κ remains supercompact and is also the least strongly compact cardinal.*
2. *If κ is strongly compact, then there is a forcing extension in which κ remains strongly compact and is also the least measurable cardinal.*

It is notable that (1) and (2) are incompatible, as the least supercompact is not the least measurable.

"Identity crisis", named by Magidor, describes this particular phenomenon that the least strongly compact cardinal suffers.

Magidor's question

> *A. Are supercompact cardinals and strongly compact cardinals eqalconsistent, i.e., does the consistency of strongly compact cardinal imply the consistency of supercompact cardinal? The informed guess is that the answer is "'no".*
> *B. What about the second (third, ...) strongly compact; can it be the second, (third, ...) measurable?"*
>
> *—Magidor[1976], pp33-34*

Both questions are currently widely open. Question A is related to the inner model of supercompact cardinal, which is one of the most prominent projects in large cardinal theory. The best result to question B is the following:

Theorem (Kimchi-Magidor)

Suppose n is finite, If $\kappa_0, \ldots, \kappa_{n-1}$ are supercompact, then there is a forcing extension in which any κ_i remains strongly compact and is also the i-th measurable cardinal.

We remark that any model positively answering B would be highly pathetic, comparing to "well-structured" models of ZFC. It is evident that strongly compact is equal to supercompact in any such "well-structured" model.

Theorem (Goldberg)

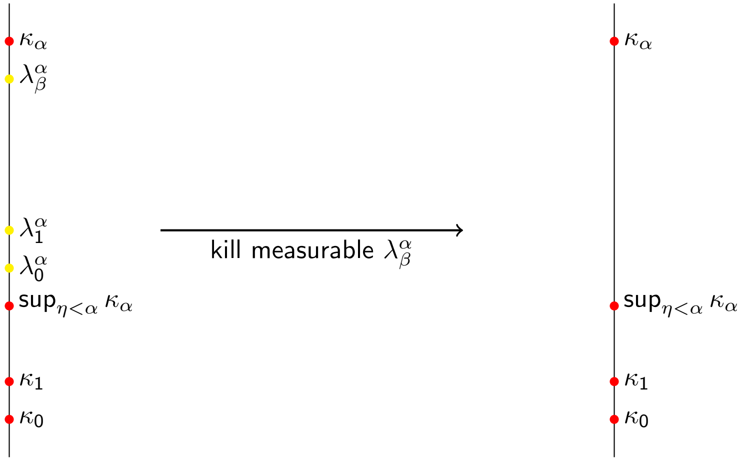
Assume ultrapower axiom (UA) holds, then the first n-many strongly compact is identical to the the first n-many supercompact.

UA holds true in any known core model.

Theorem

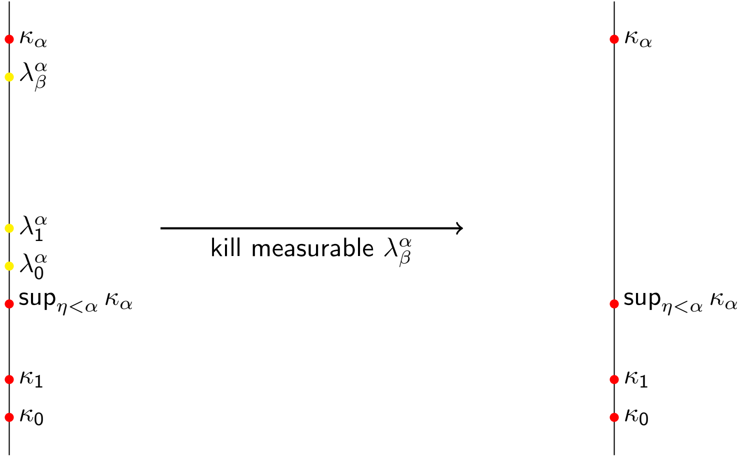
If $\kappa_0, \kappa_1, \ldots$ are supercompact and no measurable cardinals above them, then there is a forcing extension in which any κ_n remains strongly compact and is also the n-th measurable cardinal.

Basic idea for Kimchi-Magidor



For each λ^α_β, Kimchi-Magidor apply non-reflection stationary forcing to kill measurability of λ^α_β. They also shows that the class of measurable cardinals in the extension is exactly the class consisting of all κ_α.

Basic idea for Kimchi-Magidor



For each λ^α_β, Kimchi-Magidor apply non-reflection stationary forcing to kill measurability of λ^α_β. The upshot is to verify that all κ_α remain strongly compact. Kimchi-Magidor complete the verification when there are only finite many α.

Lifting argument

We use Silver's lifting criterion to show that some κ is a large cardinal.

Suppose $j : V \to M$ witness that κ is a large cardinal.

Suppose G is a P-generic over V.

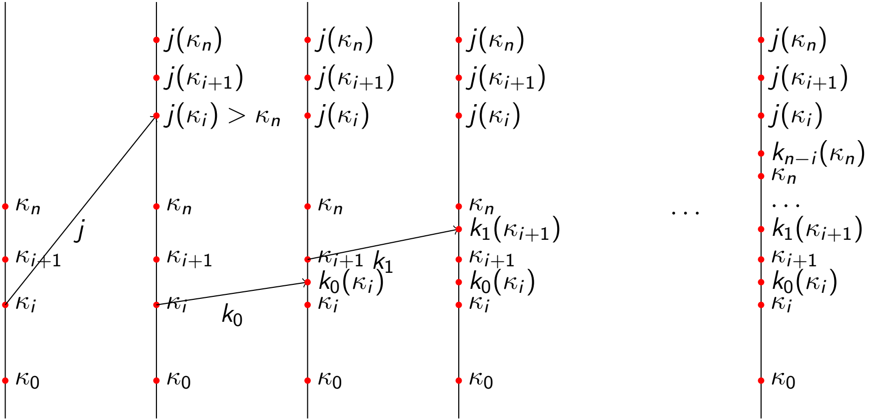
We say j can be lifted to $V[G]$ if there is a $j(P)$ generic over H in $V[G]$ such that $j[G] \subseteq H$.

And for any such H, there is an unique elementary embedding $j^+ : V[G] \to M[H]$ with $k^+(G) = H$ and $k^+ \restriction V = j$.

j^+ witness that κ is a large cardinal in $V[G]$. (In our case j^+ is strongly compact)

Lifting argument for finite many strongly compact

Assume there are only finite many κ_i, Kimchi-Magidor manage to verify the following strongly compact embedding can be lifted:



$$k = k_{n-i} \circ k_{n-i-1} \circ \dots \circ k_0 \circ j$$

There are two main obstacles for extending Kimchi-Magidor's proof to infinite case.

1. For technical reason, we need to assume there is no measurable cardinal above κ_n.
2. It is difficult to construct a lifting generic at $k(\sup_{n<\omega} \kappa_\beta)$ for limit κ.

Swallowed pair of extenders

The main new tool in our proof is the swallowed pair of extenders. We can construct such kind of pair using supercompactness.

Definition

For regular cardinals $\kappa < \lambda$, we say a pair of extender (E, F) is a (κ, λ)-swallowed pair if

- E is a (κ, η_0)-extender for some η_0.
- F is a (κ, η_1)-extender for some η_1.
- $j_E(V_\lambda) = (V_\alpha)^{j_F(V)}$ for some $\alpha < j_F(\kappa)$.
- κ is the common critical point of both j_E and j_F.

Swallowed pair is used for the construction of lifted generic over a long extender type embedding.

Thank you for your attention!