A transfinite hierarchy of low$_2$ c.e. degrees

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Capturing common dynamics

High degrees

Some degree classes capture the combinatorics that are common to several computable constructions. For example:

Theorem (Martin)

A c.e. degree is high iff. . .

1. *it contains a maximal set;*
2. *it contains a hyperhypersimple set.*

The key combinatorial fact about high degrees:

Theorem (Martin)

A c.e. degree is high if and only if it contains a function which dominates all computable functions.

Such a function is hard to compute because it grows too quickly.

ANR degrees

Downey, Jockusch and Stob introduced the *array nonrecursive* degrees, to similarly capture a class of constructions.

Theorem

A c.e. degree is array nonrecursive iff. . .

1. *it is the degree of a perfect thin Π^0_1 class [Cholak, Coles, Downey, Herrmann];*
2. *it bounds a separating class avoiding $\mathbf{0}'$ [Downey, Jocksuch, Stob];*
3. *it contains a c.e. set with maximal Kolmogorov complexity [Kummer];*
4. *it has effective packing dimension 1 [Downey,G];*
5. *it contains left-c.e. reals with no common cl-upper bound [Barmpalias, Downey, G];*
6. *it contains a set not reducible to $\emptyset'$ with tiny use [Franklin,G,Stephan,Wu].*

ANR degrees and computable approximations

Lemma (Shoenfield)

A function $f : \mathbb{N} \to \mathbb{N}$ is Δ^0_2 if and only if it has a computable approximation: a uniformly computable sequence $\langle f_s \rangle$ of functions converging to f: for all n, for almost all s, $f_s(n) = f(n)$.

Definition

The *mind-change function* of a computable approximation $\langle f_s \rangle$ is

$$m_{\langle f_s \rangle}(n) = \# \{ s \, : \, f_{s+1}(n) \neq f_s(n) \} .$$

Theorem (Downey,Jockusch,Stob)

A c.e. degree $\mathbf{d}$ is array recursive if and only if every function $f \in \mathbf{d}$ has a computable approximation $\langle f_s \rangle$ with mind-change function bounded by the identity.

Thus, the degree is powerful because it contains a function which is difficult to approximate.

A non-uniform version

Recall that $f \leqslant_{\mathrm{wtt}} A$ if f is Turing reducible to A with use bounded by some computable function.

Fact

A function f is wtt-reducible to $\varnothing'$ iff it has a computable approximation $\langle f_s \rangle$ with mind-change function bounded by some computable function.

Definition

A c.e. degree $\mathbf{d}$ is *wtt-AR* (temporary name) if every $f \in \mathbf{d}$ is $\leqslant_{\mathrm{wtt}} \varnothing'$.

A c.e. degree is *wtt-ANR* if it is not wtt-AR.

wtt-ANR combinatorics

Theorem

A c.e. degree is wtt-ANR iff...

1. *it bounds a critical triple in the c.e. degrees [Donwey,G,Weber];*
2. *it computes a left-c.e. real r with no c.e. presentation computing r [Downey,G];*
3. *it computes a computably-finite-random left-c.e. sequence [Brodhead,Downey,Ng];*
4. *it computes a left-c.e. real, not cl-reducible to a complex left-c.e. sequence [Ambos-Spies,Fang,Losert,Merkle,Monath];*
5. *it contains a set not wtt-reducible to a ranked / hypersimple set [Barmpalias,Downey,G];*
6. *it bounds an ω-change generic sequence [McInerney].*

Generalising, using Ershov’s hierarchy

Ershov's hierarchy

Recall: A set is...

- d.c.e., if it is $A_1 - A_0$, where A_i are c.e.;
- 3-c.e., if it is $(A_2 - A_1) \cup A_0$, where A_i are c.e.;
- ...

(we may assume $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$).

Ershov defined the *α-c.e.* sets, where α is a *notation* for computable ordinal: the sets in this class are of the form

$$\dots \cup (A_\beta - A_{\beta-1}) \cup (A_{\beta-2} - A_{\beta-3}) \cup \dots$$

Where $A_0 \subseteq A_1 \subseteq \dots \subseteq A_\beta \subseteq \dots$ ($\beta < \alpha$) are uniformly c.e.

Ershov's hierarchy: Δ^{-1}_α

- Let Σ^{-1}_α denote the collection of α-c.e. sets;
- Π^{-1}_α denotes the dual class (complements of sets in Σ^{-1}_α);
- $\Delta^{-1}_\alpha = \Sigma^{-1}_\alpha \cap \Pi^{-1}_\alpha$.

Proposition (Ershov)

A set A is Δ^{-1}_α iff it has a computable approximation $\langle A_s \rangle$ for which there are uniformly computable $\langle o_s \rangle \colon \mathbb{N} \to \alpha$ such that:

- *For all n and s, $o_{s+1}(n) \leqslant o_s(n)$;*
- *If $A_{s+1}(n) \neq A_s(n)$ then $o_{s+1}(n) < o_s(n)$.*

The sequence $\langle o_s(n) \rangle_s$ "counts down" α and witnesses that $\langle f_s(n) \rangle_s$ stabilizes. The longer α is, the more "breathing room" there is for changes, and so more complicated functions can be approximated.

α-c.a. functions

We can extend the notion to functions $f\colon \mathbb{N} \to \mathbb{N}$ in the same way:

Definition

An *α-computable approximation* is a uniformly computable sequence $\langle f_s, o_s \rangle$ of functions $f_s\colon \mathbb{N} \to \mathbb{N}$ and $o_s\colon \mathbb{N} \to \alpha$ satisfying

- For all n and s, $o_{s+1}(n) \leqslant o_s(n)$;
- If $f_{s+1}(n) \neq f_s(n)$ then $o_{s+1}(n) < o_s(n)$.

The function being approximated is $\lim_s f_s$.

Definition

A function $f\colon \mathbb{N} \to \mathbb{N}$ is *α-computably approximable* if it has an α-c.a. approximation.

We use the abbreviation *α-c.a.*

A caveat

Unlike for iterations of the Turing jump, *notations matter*:

Proposition (Ershov)

For every Δ^0_2 function f there is some notation b for ω^2 such that f is b-c.a.

The complexity of f is coded not into the approximation, but into the copy of ω^2.

To define α-c.a. for computable *ordinals* α we restrict ourselves to particularly nice notations, which are all computably comparable. This cannot be done for all computable ordinals. For $\alpha \leqslant \epsilon_0$ it suffices to require a computable Cantor normal form.

Henceforth, for simplicity, all ordinals are $\leqslant \epsilon_0$.

Totally α-c.a. degrees

Definition

A c.e. Turing degree $\mathbf{d}$ is *totally α-c.a.* if every $f \in \mathbf{d}$ is α-c.a.

A function f is $\leqslant_{\mathrm{wtt}} \varnothing'$ if and only if it is ω-c.a., so:

- A c.e. degree is wtt-AR iff it is totally ω-c.a.

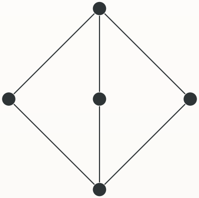
More dynamics

Theorem (Downey,G)

The following are equivalent for a c.e. degree **d***:*

1. **d** *bounds a copy of the 1-3-1 lattice in the c.e. degrees;*
2. *there is some $f \in$* **d** *which is not ω^n-c.a. for any n.*

We call such degrees not totally $< \omega^\omega$-c.a.



Even more dynamics

Theorem (Downey,G)

1. *An m-topped degree is not totally $< \omega^\omega$-c.a.;*
2. *There is an m-topped degree which is totally ω^ω-c.a.*

Here there is an extra complication due to the fact that *m*-topped degrees cannot be low.

Outside the c.e. degrees

Downey, Jockusch and Stob extended the notion of array noncomputability to the general Turing degrees, using domination.

- A Δ^0_2 Turing degree $\mathbf{d}$ is low$_2$ iff some Δ^0_2 function f dominates all functions in $\mathbf{d}$ [Martin].
- A c.e. degree is array recursive iff some ω-c.a. function dominates all functions in $\mathbf{d}$ [Downey,Jockusch,Stob].
- A c.e. degree is totally α-c.a. iff every $f \in \mathbf{d}$ is dominated by some α-c.a. function.

McInerney and Ng related these notions to "α-change" genericity.

Joins

These classes can give further information about the Sacks splitting theorem.

Theorem (Ambos-Spies,Downey,Monath,Ng)

Every c.e. set can be split into two sets, both of totally ω^2-c.a. degree.

Theorem (Downey,Ng)

There is a c.e. degree which is not the join of two totally ω-c.a. degrees.

The hierarchy

Lowness

What can we say about the hierarchy of totally α-c.a. degrees?

Vis-a-vis jump classes:

- Each level contains both low and nonlow degrees.
- Every totally α-c.a. degree is low_2.
- Every superlow degree is AR, so totally ω-c.a.

Collapse

Definition

- A function is *properly α-c.a.* if it is α-c.a., but not β-c.a. for any $\beta < \alpha$.
- A degree is *properly totally α-c.a.* if it is totally α-c.a., but not totally β-c.a. for any $\beta < \alpha$.

A diagonalisation argument shows:

- For every α, there is a properly α-c.a. function.

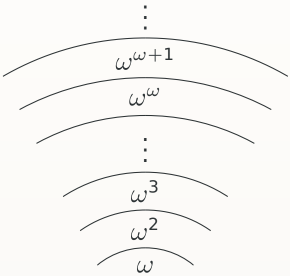
Not so for degrees.

Collapse

Theorem (Downey,G)

There is a properly totally α-c.a. degree iff α is closed under ordinal addition.

Recall that an ordinal is closed under addition iff it is a power of ω.



Definability

Recall:

Theorem (Downey,G,Weber)

A c.e. degree is totally ω-c.a. iff it does not bound a critical triple in the c.e. degrees.

Corollary

The totally ω-c.a. degrees are definable in the c.e. degrees.

Similarly:

Proposition (Downey,G)

The totally $< \omega^\omega$-c.a. degrees are definable in the c.e. degrees.

Question

Are other levels of the hierarchy definable?

This is related to lattice embeddings into the c.e. degrees, which have been studied widely. There are some recent results by Cholak and Ko, and by Ambos-Spies et al.

Maximality

Theorem (Downey,G)

For every α (a power of ω), there are maximal totally α-c.a. degrees: degrees $\mathbf{d}$ which are totally α-c.a., but no $\mathbf{a} > \mathbf{d}$ is totally α-c.a.

It is unusual to find maximal elements of subclasses of the c.e. degrees:

- No jump classes have maximal elements;
- There are no maximal cappable degrees.

A previous known example:

- Maximal contiguous degrees [Cholak,Downey,Walk].

The maximal totally ω-c.a. degrees form a definable antichain in the c.e. degrees.

Remark

There are no maximal totally $< \omega^\omega$-c.a. degrees. So there are maximal degrees with respect to not bounding critical triples, but not with respect to not bounding 1-3-1.

Collapse in upper cones

Suppose that **a** is totally α-c.a., and that $\beta > \alpha$, both powers of ω. We know that there are totally β-c.a. degrees which are not totally α-c.a.
Can we find such a degree above **a**?

Theorem (Arthur,Downey,G)

If $\beta \geqslant \alpha^\omega$ then every totally α-c.a. degree is bounded by a properly totally β-c.a. degree.

The proof of the theorem uses a basic fact of ordinal arithmetic:

- If $\beta \geqslant \alpha^\omega$ then $\alpha \cdot \beta = \beta$.

What if $\beta \in [\alpha, \alpha^\omega)$?

Collapse in upper cones

Theorem (Arthur,Downey,G)

Every totally α-c.a. degree **a** *is bounded by a totally α^4-c.a. degree which is not totally α-c.a.*

For the simplest test case, suppose $\alpha = \omega$.

Theorem (Li Ling Ko)

Every totally ω-c.a. degree is bounded by a properly totally ω^2-c.a. degree.

The construction is non-uniform. The general case is unclear.

Maximality and collapse

Theorem (Downey,G)

Every totally α-c.a. degree lies strictly below a degree which is totally $(\alpha \cdot \omega)$-c.a.

(Note that $\alpha \cdot \omega$ is the next level; if $\alpha = \omega^\beta$ then $\alpha \cdot \omega = \omega^{\beta+1}$).

Corollary

No degree at level α is maximal for higher levels.

Corollary

A maximal totally α-c.a. degree is properly totally α-c.a.

Thus, a question closely related to collapse in upper cones is:

Question

For which pairs $\beta \geqslant \alpha$ is every totally α-c.a. degree bounded by a maximal totally β-c.a. degree?

Maximality and collapse: some results

Question
For which pairs $\beta \geqslant \alpha$ is every totally α-c.a. degree bounded by a maximal totally β-c.a. degree?

For example, the $\beta \geqslant \alpha^\omega$ result mentioned above follows from:

Theorem (Arthur,Downey,G)
If $\beta \geqslant \alpha^\omega$ then every totally α-c.a. degree is bounded by a maximal totally β-c.a. degree.

On the other hand:

Theorem (Arthur,Downey,G)
For every α there is a totally α-c.a. degree which is not bounded by any maximal totally α-c.a. degree.

What about $\beta \in (\alpha, \alpha^\omega)$?

The most recent result

Test case:

Question

Is every totally ω-c.a. degree bounded by a maximal totally ω^2-c.a. degree?

Theorem (Downey,G,Hammatt)

There is no uniform way to obtain such a degree.

Thank you