Differential Chow Forms and Differential Chow Varieties: An Overview

Wei Li

KLMM, Academy of Mathematics and Systems Science,
Chinese Academy of Sciences

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Jointly with X.S. Gao, C.M. Yuan, T. Scanlon, J. Freitag, Y.H. Li

Differential Algebraic Geometry

Three possible languages to study differential equations: the classical language, the geometric language and the language of **differential algebra***.—(Yuri Manin, 1979)*

- A subject, founded by Ritt and Kolchin (1950-70s), aims to study **algebraic differential equations** based on algebraic geometry and commutative algebra.

- Basic object: **differential variety** (i.e., the solution set of a system of algebraic differential equations)

- Main Problems: algebraic theory and algorithms around differential equation solving.

Relationship with Model Theory

The origins of model theory and differential algebra may be starkly different in character, but in recent decades large parts of these subjects have developed symbiotically. (Scanlon, 2002)

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- Differential closed fields serve as proving and testing grounds for pure model theory.

  "the least misleading example for many model-theoretic phenomena" (Sacks, 1972).

Outline

- Motivation: Algebraic Chow forms and Chow varieties
- Ordinary differential Chow forms and Chow varieties
- Partial differential Chow forms and a type of partial differential Chow varieties
- Summary and Problems

Motivation

Chow-van der Waerden (Math. Ann., 1937) pointed out:

> It is principally important to represent geometrical objects by coordinates. Once this has been done for a specific kind of objects G, then it makes sense to speak of an algebraic manifold or an algebraic system of objects G, and to apply the whole theory of algebraic manifolds (decomposition into irreducible components, notion of dimension, notion of general elements in an irreducible manifold). It is desirable to provide the set of objects G with the structure of an algebraic variety (eventually, after a certain compactification), thus to characterise G by algebraic equations in the coordinates.

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Simple examples:

- Points: Cartesian coordinates
- Lines in $\mathbb{P}^3$: Plücker coordinates
- Linear spaces: Grassmann coordinates

Example: Plücker coordinates

Let V be a line in $\mathbb{P}^3$ defined by

$$\begin{cases} a_{00}x_0 + a_{01}x_1 + a_{02}x_2 + a_{03}x_3 = 0 \\ a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0. \end{cases}$$

- Plücker coordinates of V: $(p^{01}, p^{02}, p^{03}, p^{12}, p^{13}, p^{23}) \subset \mathbb{P}^5$, where

$$p^{ij} = a_{0i}a_{1j} - a_{0j}a_{1i}.$$

- **The variety of lines**: $p^{01}p^{23} + p^{02}p^{31} + p^{03}p^{12} = 0.$

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Algebraic varieties: Chow coordinates and Chow varieties

(was developed based on the theory of **algebraic Chow forms**)

Algebraic Chow forms (Chow, 1937)

- Let $\mathbf{V} = \sum_{i=1}^{l} s_i V_i$ be a d-cycle in $\mathbb{P}^n$ $(\dim(V_i) = d)$.

  Let $L_i = u_{i0}y_0 + u_{i1}y_1 + \cdots + u_{in}y_n \, (i = 0, \ldots, d)$: hyperplanes

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- Chow coordinates of $\mathbf{V}$: coefficients $(c_\omega) \in \mathbb{P}^N$.
- Degree of $\mathbf{V}$: $\deg(\mathbf{V}) = \deg(F, (u_{ij})_{j=0}^{n}) = g$.

  Poisson-product formula: $F = A\prod_{\tau=1}^{\mathbf{g}}(u_{00}\xi_{\tau 0} + \cdots + u_{0n}\xi_{\tau n})$.

Chow varieties

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Examples of applications:

- Chern classes on algebraic varieties with arbitrary singularities (Wu, 1984);
- Counting problems in geometry;
- Arakelov theory and diophantine applications.

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- By GDIT, there exists a unique $F(u_{ij}) = \sum_\omega c_\omega M_\omega(u_{ij})$ s.t.

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Example: $V = \mathbb{V}(y' - y) \subset \mathbb{A}^1$

- δ-Chow form of V: $F = u_{00}' u_{01} - u_{00} u_{01}' - u_{00} u_{01}$;
- δ-Chow coordinate of V: $(1, -1, -1, 0, 0, 0, 0, 0, 0, 0) \in \mathbb{P}^9$.

New invariants of differential varieties

Properties of δ-Chow form $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$:

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- The g points $(\xi_{\tau 1}, \ldots, \xi_{\tau n})$ are the only common points of
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The above V is called of index (d, h, g, m).

Ordinary Diff Chow variety

- δ-cycle: $\mathbf{V} = \sum_i s_i V_i \quad (V_i \subseteq \mathbb{A}^n$ irr. of dim d, order $h)$.
- $\mathcal{C}_{(d,h,g,m)} \triangleq \{\delta\text{-Chow coordinates of } \mathbf{V} \text{ of index}\,(d,h,g,m)\}$.

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- δ-Chow varieties $\mathcal{C}_{(d,h,g,m)}$ exist. (Freitag-Li-Scanlon, 2017)
  A model-theoretical proof.

Ingredients of the proof

- $\delta\text{-chow}_n(d, h, g, m) = \big\{\text{diff cycles in } \mathbb{A}^n \text{ of index}\,(d, h, g, m)\big\}$.
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$$\delta\text{-chow}_n(d,h,g,m) \ni \textstyle\sum_{i=1}^{\ell} V_i$$

$$\Big\downarrow \phi$$

$$\bigcup_{e=D_1}^{D_2} \text{chow}_{n(h+1)}(d(h+1)+h, e) \ni \textstyle\sum_{i=1}^{\ell} B_h(V_i)$$

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Theorem. The image $\mathcal{C}$ is δ-constructible and the map ϕ identifies $\delta\text{-chow}_n(d, h, g, m)$ with $\mathcal{C}$.

Tools:

- Definability results in the model theory of ACF_0 and DCF_0;
- Moosa-Scanlon's prolongation theory of jet spaces.

Partial Differential Chow Forms and Chow Varieties

Partial differential Chow form: problem

- Let $V \subseteq \mathbb{A}^n$ be an irreducible $\Delta(=\{\delta_1, \ldots, \delta_m\})$-variety of dimension d.

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- $S := \left\{(u_{ij})_{i,j=0}^{d,n} : V \cap \mathbb{V}(L_0, \ldots, L_d) \neq \emptyset\right\}^{\mathrm{cl}}$.

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Problem: There may not exist a single F s.t. $S = \mathbb{V}(\mathrm{sat}(F))$.

Non-Example. Let $m = 2$, $V = \mathbb{V}(\delta_1(y), \delta_2(y)) \subset \mathbb{A}^1$. Here $S = \mathbb{V}(\mathrm{sat}(u_{01}\delta_1(u_{00}) - u_{00}\delta_1(u_{01}), u_{01}\delta_2(u_{00}) - u_{00}\delta_2(u_{01}))$.

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Def. If $S = \mathbb{V}(\mathrm{sat}(F))$, then call F the Δ-Chow form of V.

For which varieties V, the Δ-Chow form of V exists?

A type of Δ-Chow varieties (Li, 2020)

Theorem. If $\omega_V(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$, then the Δ-Chow form F of V exists. Moreover, $\mathrm{ord}(F) = s$ and

- F is Δ-homogenous of degree r in the coefficients of each L_i.
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Conjecture. The above condition is also a necessary one.

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Let $\Delta\text{-}\mathrm{Chow}_n(d, s, r)$ be the set of all irr. Δ-varieties $V \subseteq \mathbb{A}^n$ with $\omega_V(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$ and Δ-deg$(V) = r$.

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Theorem. $\Delta\text{-}\mathrm{Chow}_n(d, s, r)$ is a Δ-constructible set.

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The theory of ordinary differential Chow forms and differential Chow varieties is established, and limited results are obtained in the partial differential case.

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Further Problems:

- Constructive proof for the existence of δ-Chow varieties;
- Conjecture on the the existence of Δ-Chow forms;
- How to give coordinate representations and parameter space for general Δ-varieties?

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Thanks!