Subclasses of effective supermartingales: completeness phenomenon

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Introduction

- What is randomness?
- Randomness $\Leftrightarrow$ "No pattern".
- Strings with some "pattern": 010101010101, 011000111100000111111.

Introduction

- Effective randomness ⇔ No effective pattern.
- Effective pattern: a sequence $(V_n \subseteq 2^{<\omega} : n \in \omega)$ of uniformly c.e. sets (with $[V_n] \supseteq [V_{n+1}]$) such that $m(V_n) \leq 2^{-n}$ (known as *Martin-Löf test*).

Introduction

Definition 1

A real $X \in 2^\omega$ is *Martin-Löf random* (also called *1-random*) if no Martin-Löf test $(V_n : n \in \omega)$ succeed on X. i.e., $X \notin \bigcap_n [V_n]$.

- Many definitions of effective randomness turn out to be equivalent (to 1-randomness).
- For example, X if 1-random iff there is no left-c.e. supermartingale M succeeding on X (i.e., $\limsup_n M(X \upharpoonright n) < \infty$).
- Here a *left-c.e. supermartingale* is a non decreasing computable array $(M[t] : t \in \omega)$ of supermartingales such that $\lim_{t \to \infty} M[t](\sigma) = M(\sigma)$ exists for all $\sigma \in 2^{<\omega}$.

Introduction

- Unfortunately all definitions of 1-randomness concern c.e.ness, which is dissatisfactory since it is supposed to be an effective randomness notion. Numerous definitions that try not to use c.e.ness are given such as:
    1. Schnorr randomness: the reals on which no Schnorr test succeed (a Schnorr test is a Martin-Löf test with $m(V_n)$ being computable);
    2. Kurtz randomness: the reals that cannot be contained in any measure 0 effectively closed subset of 2^ω;
    3. computable randomness: the reals on which no computable martingale succeed.
- But none of them are as strong as 1-randomness (1-randomness implies them but not vice versa).

Introduction

Is there a complexity notion weaker than left-c.e.ness yet makes the supermartingales (of that complexity) define 1-randomness.

Question 2

Or is there a class of left-c.e. supermartingales whose behaviour is somewhat “predictable” defining 1-randomness.

kastergale

- For a computable martingale M, we could know (computably) whether $M(\sigma 1) \geq M(\sigma 0)$.
- We say M is *i-sided at* σ if

$$M(\sigma i) \geq M(\sigma^\frown(1-i)).$$

- We say M is *p-sided* if for every $\sigma \in dom(p)$, M is $p(\sigma)$-sided at σ, and for every $\sigma \notin dom(p)$, M is both 0-sided, 1-sided at σ.

kastergale

Definition 3 (kastergale)

For left-c.e. supermartingale M, we say M is *partially-computably-sided* (known as *kastergale*) iff:

$$\text{for some partial computable function } p,\ M[t] \text{ is } p[t]\text{-sided.}$$

i.e., For each $\sigma \in 2^{<\omega}$, M has only one chance to decide its sidedness at σ and before it makes that decision, it has to be both $0, 1$-sided at σ.

muchgale

Definition 4 (muchgale)

A supermartingale M is *(l, i)-betting* if for every σ such that $|\sigma| \equiv i \mod (l)$, we have $M(\sigma) \geq \max\{M(\sigma 0), M(\sigma 1)\}$. i.e., M does not bet at certain steps. A *muchgale* is a left-c.e. supermartingale that is (l, i)-betting for some l, i.

Other subclasses

- Integer-valued supermartingales;
- Fine and coarse granularity;

Questions and known results

- Kasterman wondered if kastergales define 1-randomness (i.e., whether for every non-1-random real X there is a kastergale succeeding on X) [Downey, 2012];
- Hitchock asked the same question with respect to a subclass of kastergale where the biased proportion $M(\sigma i)/M(\sigma)$ is Σ^0_1 function;
- Barmpalias, Fang and Lewis-Pye [Barmpalias et al., 2020] considered single-sided (p-sided with $p \equiv i$ for some $i \in 2$) left-c.e. supermartingales whose bias is non decreasing and showed that they do not define 1-randomness.
- Muchnick [Muchnik, 2009] considered $(2, i)$-betting left-c.e. supermartingales and showed that they do not define 1-randomness.

Another view point

- One of the most famous open question in computability randomness theory is that whether KL-randomness is equivalent to 1-randomness.
- This is the same as asking whether the class of betting strategies defining KL-randomness succeed on all non-1-random reals.
- However, this class is not a subclass of left-c.e. supermartingales, therefore our method cannot be directly applied.

Conclusion

Theorem 5 ([Barmpalias and Liu, 2021])

The union of kastergales and muchgales does not define 1-randomness. i.e., there is a non-1-random real X on which no kastergale or muchgale succeed.

Conclusion

Our analysis shows that

If a reasonable subclass of left-c.e. supermartingales defines 1-randomness, it almost means a single member of that class can do so. (2.1)

Formalize (2.1)

- A class of *supermartingale-approximations* is a set $\mathcal{M}$ of supermartingale sequences $M[\leqslant t] = (M[0], \cdots, M[t])$.
- $\mathcal{M}$ is *non decreasing* iff: $M[t]$ dominates $M[t-1]$;
- $\mathcal{M}$ is *effective* iff: $M[\leqslant t] \in \mathcal{M}$ is decidable.
- An *$\mathcal{M}$-gale* is: a ω-sequence $M[< \omega]$ such that $M[\leqslant t] \in \mathcal{M}$ for all $t \in \omega$ and $\lim_{t\to\infty} M[t](\sigma)$ exists for all $\sigma \in 2^{<\omega}$.

Formalize (2.1)

- We say $\mathcal{M}$ is *homogeneous* iff, roughly speaking, looking at $\mathcal{M}$ on a cone $[\rho]^{\preceq}$ is the same as that on $[\emptyset]^{\preceq}$.
- Homogeneous class:
  kastergales;
  given l, $\{(l, i)\text{-betting supermartingales} : i < l\}$;
  muchgale.
- In (2.1), by "reasonalbe", we mean homogeneous and effective.

A game

Whether computable $\mathcal{M}$-gales define 1-randomness $\hookrightarrow$

Whether Alice (controlling the Martin-Löf test) wins against Baby (controlling members of $\mathcal{M}$) in the following game.

A game

The finite version of this game:

Definition 6 ((c, n, k)-$\mathcal{M}$-game)

At each round $t \in \omega$:
Alice: enumerates $\sigma \in 2^n$;
Baby: presents $M_j[t]$ (for each $j < k$) such that:

- $\sum_j M_j[t](\hat{\sigma}) \geq 1$ for some $\hat{\sigma} \preceq \sigma$ (for all $\sigma \in A[t]$);
- $M_j[\leqslant t] \in \mathcal{M}$ for all $j < k$.

Alice wins if: $\sum_j M_j[t](\emptyset) \geq c$.

Let A denote the set of σ Alice enumerates when she wins.

A game

- Roughly speaking, if Alice has a winning strategy for (c, n, k)-$\mathcal{M}$-game with an arbitrary small cost $m(A)$, then $\mathcal{M}$ does not define 1-randomness.
- Let$\mathcal{M} = \cup_l \mathcal{M}_l$ where $\mathcal{M}_l \subseteq \mathcal{M}_{l+1}$ is uniformly effective, scale-closed, non decreasing and homogeneous.

Claim 7

If for every $l, k \in \omega, \varepsilon > 0, c < 1$, Alice has a winning strategy for (c, n, k)-$\mathcal{M}_l$-game (for some n) such that $m(A) \leq \varepsilon$, then computable $\mathcal{M}$-gales do not define 1-randomness.

The constant game

Let $a, \Delta, \delta > 0, n, k \in \omega$:

Definition 8 (constant $(a, \Delta, \delta, n, k)$-$\mathcal{M}$-game)

At each round $t \in \omega$:
Alice: $\sigma \in 2^n$,
Baby: $M_j[t]$ such that:

- $\sum_j M_j[t](\sigma) \geq 1$ (for all $\sigma \in A[t]$);
- $M_j[\leqslant t] \in \mathcal{M}$ for all $j < k$.
- $\sum_j M_j[t](\rho) \leq 1 + \delta$ for all $\rho \in 2^{\leq n}$.

Alice wins if:

- (type-(a)) $1 - \sum_j M_j[t](\emptyset) \leq (1 - m(A[t]))/a$; or
- (type-(b)) for some $\sigma_0, \sigma_1 \in A[t]$, $||\vec{M}[t](\sigma_0) - \vec{M}[t](\sigma_1)||_1 \geq \Delta$

constant $\mathcal{M}$-game vs $\mathcal{M}$-game

- "$\sum_j M_j[t](\sigma) \geq 1$" vs
  "$\sum_j M_j[t](\hat{\sigma}) \geq 1$ for some $\hat{\sigma} \preceq \sigma$";
- $\sum_j M_j[t](\rho) \leq 1 + \delta$;
- dynamic winning criterion "$1 - \sum_j M_j[t](\emptyset) \leq (1 - m(A[t]))/a$" vs
  "$\sum_j M_j[t](\emptyset) \geq c$"
- for some $\sigma_0, \sigma_1 \in A[t]$, $||\vec{M}[t](\sigma_0) - \vec{M}[t](\sigma_1)||_1 \geq \Delta$

Reduce to constant game

- Roughly speaking, if Alice could win the constant $\mathcal{M}$-game (for $k = 1$) with $m(A) < 1$, then she could win the $\mathcal{M}$-game (for all k) with an arbitrary small $m(A)$.
- Let$\mathcal{M}$ be non decreasing and homogeneous.

Claim 9

If for every $a > 0$, there are $\Delta, \delta > 0, n \in \omega$ such that Alice has a winning strategy for the constant $(a, \Delta, \delta, n, 1)$-$\mathcal{M}$-game with $m(A) < 1$, then for every $\varepsilon > 0, c < 1, k \in \omega$ there is an n such that Alice has a winning strategy for (c, n, k)-$\mathcal{M}$-game such that $m(A) \leq \varepsilon$.

Reduce to constant game

Proof.

See [Barmpalias and Liu, 2021].
section 2.1-2.2 (dynamic winning criterion),
section 2.3 (restricting Baby's action),
section 4.2 (type-(b) winning criterion),
section 4.3 (reduce to $k = 1$). □

Reduce to constant game

- For kastergale or (l, i)-betting supermartingale-approximation, it's easy to win the constant game (for $k = 1$), thus Theorem 5 follows.
- Winning the constant game (for $k = 1$) is the only part of the proof where we take advantage of sidedness and (l, i)-betting.

Completeness phenomenon

- Moreover, if $\mathcal{M}$ could define 1-randomness, then (for some $a > 0$, for every $\Delta, \delta > 0$, every $n \in \omega$) Alice does not have a winning strategy for the constant $(a, \Delta, \delta, n, 1)$-$\mathcal{M}$-game so that $m(A) < 1$.
- This almost means that a single member of $\mathcal{M}$ (the one Baby used against Alice) could define 1-randomness.
- With that said, this is not a concrete proof of (2.1), but a strong evidence.

A more efficient winning strategy

For $\mathcal{M} = \{(2, i)\text{-betting supermartingale-approximation}\}$, Alice can win the (c, n, k)-$\mathcal{M}$-game (with $c = 1, n = 2$) with a cost $m(A) \leq \frac{3}{4}$. Thus

Theorem 10 ([Barmpalias and Liu, 2021])

There is a real $X \in 2^\omega$ on which no $(2, i)$-betting left-c.e. supermartingale succeeds for all $i < 2$ such that $\dim_H(X) \leq 1 - \frac{1}{2}\log_2(4/3)$.

Given a subclass $\mathcal{M}$ of left-c.e. supermartingales and $d \geq 0$,

Question 11

Is there a real X with $dim_H(X) \leq d$ such that there is no member of $\mathcal{M}$ succeeding on X.

Question 12

Is there a winning strategy of Alice on the (c, n, k)-$\mathcal{M}$-game (when n is sufficiently large) such that $m(A) \leq \exp(-O(1)n)$?

Many thanks

Is there any question?

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