Countable sections for actions of locally compact groups

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Introduction

In this talk I will discuss some recent progress on an old, c. 1992, and still open problem, concerning the descriptive set theory of dynamics of locally compact groups. I will start with some preliminaries and historical background.

To avoid excessive repetition, in this talk a **locally compact group** will be always **second countable** (and therefore a Polish group).

A **standard Borel space** is a Polish space with the associated Borel structure. Given a Borel action $(g,x) \in G \times X \mapsto g \cdot x \in X$ of a Polish group G on a standard Borel space X, we denote by E_G^X the associated **orbit equivalence relation**

$$xE_G^X y \iff \exists g \in G(g \cdot x = y).$$

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For a Borel action of a Polish group G on a standard Borel space X **a (complete) Borel section** *is a Borel subset $S \subseteq X$ that meets every orbit. Such a section is* **countable** *if it meets every orbit in a countable (non-empty) set.*

Definition

A Borel section S is **lacunary** *if there is an open nbhd U of the identity of G such that for all $s \in S$, $U \cdot s \cap S = \{s\}$. In other words, any two distinct elements $s, t \in S$ in the same orbit are "far apart", i.e., $g \cdot s = t \implies g \notin U$. Since G is second countable, every lacunary section is countable.*

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Lacunarity vs essential countability

Lacunarity is closely related to existence of countable Borel sections as was recently shown by Jan Grebík.

Theorem (Grebík, 2020)

If a Polish group G acts in Borel way on a standard Borel space X, the following are equivalent:
(i) E_G^X admits a countable Borel section;
(ii) There is a countable decomposition $X = \bigsqcup_n X_n$ into G-invariant Borel sets X_n such the action if G on each X_n admits a lacunary Borel section.

Grebík has also shown that a countable decomposition is in general necessary.

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The Feldman-Hahn-Moore Theorem

In 1979 Feldman, Hahn and Moore established the existence of lacunary Borel sections for Borel actions of locally compact groups in the measure theoretic category, i.e., neglecting null sets. More precisely they proved the following:

Theorem (Feldman-Hahn-Moore, 1979)

Let a locally compact group G act in a Borel way on a standard Borel space X and let μ be a probability Borel measure on X which is quasi-invariant (i.e., if a Borel set A is μ-null and $g \in G$, then $g \cdot A$ is also μ-null). Then there is an invariant Borel set $Y \subseteq X$ with $\mu(Y) = 1$ such that the restriction of the G-action on Y admits a lacunary Borel section.

Remark

For $G = \mathbb{R}$ this result was proved earlier by Ambrose in 1941.

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The Feldman-Hahn-Moore Theorem

The proof of this result used the deep structure theory of locally compact groups arising from the solution of the Hilbert 5th Problem to reduce it to the case of Lie groups where special tools were available.

The Borel version of the Feldman-Hahn-Moore Theorem

In 1992 I proved the strengthening of the Feldman-Hahn-Theorem in the Borel category. Moreover the proof was purely descriptive set theoretic. More precisely we have the following result:

Theorem (K, 1992)

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Borel reducibility

The preceding result has an implication to the theory of reducibility of equivalence relations.

Definition

Given Borel equivalence relations E, F on standard Borel paces X, Y, resp., we say that E is **Borel reducible** *to F if there is a Borel function $T\colon X \to Y$ such that $xEy \iff T(x)FT(y)$. In this case we write $E \leq_B F$. We say that E, F are* **Borel bi-reducible** *if $E \leq_B F$ and $F \leq_B E$, in which case we write $E \sim_B F$.*

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Let a Polish group G act in a Borel way on a standard Borel space X with E_G^X Borel. Then the following are equivalent:
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We now have the following corollary of the preceding theorem:

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Let a locally compact group G act in a Borel way on a standard Borel space X. Then the orbit equivalence relation E_G^X is essentially countable.

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Countable Borel equivalence relations

The next results give a more detailed analysis of the equivalence relations induced by locally compact group actions. Below $I_{\mathbb{R}} = \mathbb{R}^2$.

Theorem (K, 1994)

Let G be a locally compact group acting in a Borel way on a standard Borel space X. Then there is a (unique) decomposition $X = A \sqcup B$ of X into G-invariant Borel sets such that $E_G^X \restriction A$ is countable and $E_G^X \restriction B \cong_B F \times I_{\mathbb{R}}$, where $F = E_G^X \restriction S$, with S a countable Borel section of $E_G^X \restriction B$.

Corollary

The map $E \mapsto E \times I_{\mathbb{R}}$ induces a bijection between countable Borel equivalence relations, up to Borel bireducibility, and equivalence relations induced by Borel actions of locally compact groups with uncountable orbits, up to Borel isomorphism.

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More on lacunary sections

Concerning lacunary sections, it is of interest to obtain in certain situations additional information about their structure. For the case of free Borel actions of $\mathbb{R}$ on standard Borel spaces, each orbit is an (affine) copy of $\mathbb{R}$, so if S is lacunary section, it makes sense to talk about the distance between consecutive members of S in the same orbit. We now have the following result that provides a purely Borel strengthening of a classical result of Rudolph in the measure theoretic context and again neglecting null sets.

Theorem (Slutsky, 2019)

Let α, β be two rationally independent positive reals. Then any free Borel action of $\mathbb{R}$ admits a Borel lacunary section such that the distance between any two consecutive points in the same orbit belongs to $\{\alpha, \beta\}$.

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Essential countability and locally compact groups

Recall from earlier slides:

Theorem (K, 1992)

Let a locally compact group G act in a Borel way on a standard Borel space X. Then the orbit equivalence relation E_G^X is essentially countable.

In the same paper the question of whether this essential countability theorem actually characterizes Polish locally compact groups has been raised.

Problem

Let G be a Polish group with the property that all the equivalence relations induced by Borel actions of G on standard Borel spaces are essentially countable. Is the group locally compact?

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Essential countability and locally compact groups

Hjorth in his 2000 book on "Classification and orbit equivalence relations" referred to this problem as "stubbornly open," and it so remains to this day. However, progress towards an affirmative answer has been made so far for certain classes of Polish groups and I will discuss this in the reminder of the talk.

Smoothness and compact groups

Before doing that however let me mention a related result that characterizes compact groups. Recall that an equivalence relation is **smooth** if it is Borel reducible to the equality relation on a standard Borel space. It is not hard to see that if G is a compact Polish group acting in a Borel way on a standard Borel space X, then E_G^X is smooth. The converse was proved by Solecki:

Theorem (Solecki, 2000)

A Polish group is compact iff every equivalence relation induced by a Borel action of the group on a standard Borel space is smooth.

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Positive answers for some classes of Polish groups

Going back to our main

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Let G be a Polish group with the property that all the equivalence relations induced by Borel actions of G on standard Borel spaces are essentially countable. Is the group locally compact?

we have the following partial results:

Theorem (Thompson, 2006)

Every group G as above must be CLI, i.e., admits a complete left-invariant metric .

However there are many CLI groups which are not locally compact. We also have an affirmative answer for the following classes of Polish groups:

(i) (Solecki, 2000) All separable Banach spaces, under addition;

(ii) (Malicki, 2016) All abelian isometry groups of separable locally compact metric spaces.

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In a recent paper (2020) of K, Malicki, Panagiotopoulos and Zielinski, a positive answer has been now obtained for **all** groups of isometries of separable locally compact metric spaces (and more) and thus in particular for all non-archimedean Polish groups. (In addition a new approach to the case of Banach spaces was developed.)

As opposed to the proofs of the earlier results and although the statement of this new result refers to the descriptive set theoretic properties of actions of Polish groups, its proof uses ergodic theoretic methods.

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Polish groups of isometries

More precisely we have the following result:

Theorem (K, Malicki, Panagiotopoulos and Zielinski, 2020)

Let L be a separable locally compact metric space and let $G = \mathrm{Iso}(L)$ be its isometry group. If all equivalence relations induced by Borel actions of G on standard Borel spaces are essentially countable, then G is locally compact.

Note here that it was proved by Gao-K in 2003 that every closed subgroup of the group of isometries of a separable locally compact metric space is also the isometry group of a separable locally compact metric space and thus the above result also applies to all such closed subgroups and in particular to all non-archimedean Polish groups.

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Sketch of the proof

We will use the following criterion:

Theorem (Feldman-Ramsay, 1985)

Consider a Polish group G and a free Borel action of G on a standard Borel space Y which admits an invariant Borel probability measure. If G is not locally compact, then E_G^Y is not essentially countable.

So assuming that G is a non-locally compact isometry group of a separable locally compact metric space, it is enough to find such an action,

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We next need the following structure theorem for the isometry groups of separable locally compact metric spaces.

Theorem (Gao-K, 2003)

Let L be a separable locally compact metric space. Then $\mathrm{Iso}(L)$ is is (up to topological group isomorphism) a closed subgroup of a group of the form

$$\prod_n (S_\infty \ltimes K_n^{\mathbb{N}}),$$

where S_∞ acts on each product group $K_n^{\mathbb{N}}$ by $(g \cdot x)_i = x_{g^{-1}(i)}$, and each K_n is a Polish locally compact group.

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Using this we next prove the following:

Proposition

Let H be an infinite dimensional separable (complex) Hilbert space and $U(H)$ its unitary group. Let L be a separable locally compact metric space and let $G = \mathrm{Iso}(L)$. Then G is is (up to topological group isomorphism) a closed subgroup of $U(H)$.

Let now (X, μ) be a standard non-atomic probability space. It is known that $U(H)$ is a closed subgroup of the Polish group $\mathrm{Aut}(X, \mu)$ of all automorphisms of (X, μ). Thus G is (up to topological group isomorphism) a closed subgroup of $\mathrm{Aut}(X, \mu)$.

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A closed subgroup G of $\mathrm{Aut}(X,\mu)$ is called **spatial** if its action on the measure algebra of (X,μ) has a spatial realization, i.e., it is induced by a Borel measure preserving action of G on (X,μ).

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Theorem (Kwiatkowska-Solecki, 2011)

Let L be a separable locally compact metric space and let $G = \mathrm{Iso}(L)$. Then G (viewed as a closed subgroup of $\mathrm{Aut}(X, \mu)$) is spatial.

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The proof finally is completed by showing the following:

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Let (X, μ) be a standard non-atomic probability space and let G be a spatial closed subgroup of $\mathrm{Aut}(X, \mu)$. Then G admits a free Borel action on some standard Borel space Y with invariant Borel probability measure and thus if G is not locally compact, E_Y^G is not essentially countable.

For the proof of this theorem one takes the action of the spatial realization. This may not be free but its (diagonal) infinite product action will be free a.e.

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The above method depends crucially on the fact the groups that we are considering are closed subgroups of $U(H)$. However it is not known if our result holds for **all** closed subgroups of $U(H)$. This seems to be the next open problem along these lines.

THANK YOU!