Normalizing Notations in the Ershov Hierarchy

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Background

- The notion of a *c.e.* set is a fundamental one in computability theory.
- In 1965, the notion of *n-c.e.* sets was first introduced by Putnam and Gold.
- In 1968 – 1970, Ershov studied a hierarchy of sets generated by the n-c.e. sets, which exhausts the whole class of Δ^0_2-sets.
- The resulting hierarchy of sets is now known as *the Ershov difference hierarchy*.

Background

The well-known Limit Lemma characterizes the Δ^0_2-sets.

Theorem (Shoenfield)

Let $A \subseteq \omega$. The following are equivalent:

(a) A *is* Δ^0_2.

(b) $A \leq_T \emptyset'$.

(c) *There is a computable function* $f : \omega \times \omega \to \{0, 1\}$ *such that for all* $x \in \omega$,

1. $f(x, 0) = 0$;
2. $|\{s : f(x, s+1) \neq f(x, s)\}| < \infty$;
3. $\lim_{s\to\infty} f(x, s) = A(x)$.

Finite levels: n-c.e. sets

Definition

A is called an n-c.e. *set if A satisfies (1), (3) and following condition*

$$(2)\ |\{s : f(x, s+1) \neq f(x, s)\}| \leq n.$$

- The classes of n-c.e. sets form the finite levels of Ershov hierarchy.
- Traditionally when n-c.e. sets were studied, the ω-c.e. sets were included also, even if strictly speaking they do not belong to the finite levels.

Definition

A is called an ω-c.e. *set if A satisfies (1), (3) and following condition*

$$(2)\ |\{s : f(x, s+1) \neq f(x, s)\}| \leq g(x)$$

for some computable function $g(x)$.

Infinite levels: Kleene's $\mathcal{O}$

The infinite levels of the Ershov hierarchy are defined using Kleene's system of ordinal notations $(\mathcal{O}, <_o)$.
The elements of $\mathcal{O} \subset \omega$ are notations, if $a \in \mathcal{O}$, then $|a|_o$ denotes the ordinal α which has notation a.

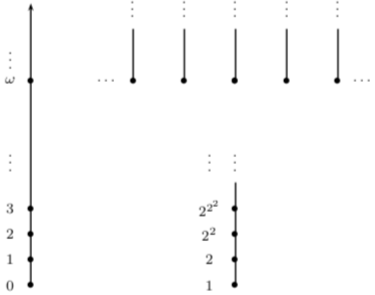
Definition

- *1 is the notation for 0.*
- *If a is a notation for α, then 2^a is a notation for $\alpha + 1$.*
  Let $b <_o 2^a$ if $b <_o a$ or $b = a$.
- *If Φ_e is a total computable function such that $|\Phi_e(n)|_o = \alpha_n$ and $\Phi_e(n) <_o \Phi_e(n+1)$ for each $n \in \omega$, then $3 \cdot 5^e$ is a notation for $\alpha = \lim_n \alpha_n$.*
  Let $b <_o 3 \cdot 5^e$ if $\exists n(b <_o \Phi_e(n))$.

The ordinals having notations in $\mathcal{O}$ are called constructive ordinals.

Infinite levels: Kleene's $\mathcal{O}$

The initial part of $\mathcal{O}$ looks like the figure below, where the vertical line to the left indicates the ordinal line:



Infinite levels: a-c.e. sets

Definition

For each $a \in \mathcal{O}$, $A \subseteq \omega$ is a-c.e. if and only if there are computable functions $f : \omega \times \omega \to \{0, 1\}$ and $o : \omega \times \omega \to \mathcal{O}$ such that for all x and s,

(1) $f(x, 0) = 0$ *and* $o(x, 0) <_o a$.

(2) $o(x, s+1) \leq_o o(x, s)$.

(3) *if* $f(x, s+1) \neq f(x, s)$ *then* $o(x, s+1) \neq o(x, s)$.

(4) $\lim_s f(x, s) = A(x)$.

We only define a-c.e. for notations $a \in \mathcal{O}$ instead of α-c.e. for constructive ordinals α. However, α-c.e. is well defined when $\alpha < \omega^2$.

Finite levels: degree theory

- Let $\mathcal{D}_n$ denote the class of n-c.e. Turing degrees and $\mathcal{D}_\omega$ the class of ω-c.e. Turing degree.
- The study of n-c.e. Turing degrees can be traced back to Cooper who showed that the degrees of difference hierarchy do not collapse.

Theorem (Cooper, 1971)

There is a degree $\mathbf{a} \in \mathcal{D}_2 - \mathcal{D}_1$.

Remark

It can be generalized to show $\mathcal{D}_1 \subsetneq \mathcal{D}_2 \subsetneq \cdots \subsetneq \mathcal{D}_\omega \subsetneq \mathcal{D}(\leq \mathbf{0}')$.

- Since 1970s, the structures $\mathcal{D}_n$ and $\mathcal{D}_\omega$ have been intensively studied.

- $\mathcal{D}_1$ is dense:

Sacks Density Theorem (Sacks, 1964)

For any $\mathbf{d}, \mathbf{b} \in \mathcal{D}_1$ *with* $\mathbf{d} < \mathbf{b}$, *there is an* $\mathbf{a} \in \mathcal{D}_1$ *such that* $\mathbf{d} < \mathbf{a} < \mathbf{b}$.

- $\mathcal{D}_n$ $(n \geq 2)$ and $\mathcal{D}_\omega$ are not dense:

The d.c.e. Nondensity Theorem (Cooper et al., 1991)

There is a $\mathbf{d} \in \mathcal{D}_2$ *with* $\mathbf{d} < \mathbf{0}'$ *such that no* $\mathbf{a} \in \mathcal{D}_\omega$ *can have* $\mathbf{d} < \mathbf{a} < \mathbf{0}'$.

- $\mathcal{D}_n$ $(n \geq 1)$ is downward dense:

Theorem (Lachlan, unpublished)

For any $\mathbf{b} \in \mathcal{D}_n$ *with* $\mathbf{b} > \mathbf{0}$, *there is an* $\mathbf{a} \in \mathcal{D}_1$ *such that* $\mathbf{0} < \mathbf{a} < \mathbf{b}$.

- $\mathcal{D}_\omega$ is not downward dense:

Theorem (Sacks, 1961)

There is a $\mathbf{b} \in \mathcal{D}_\omega$ *with* $\mathbf{b} > \mathbf{0}$ *such that no* $\mathbf{a} \in \mathcal{D}(\leq \mathbf{0}')$ *can have* $\mathbf{0} < \mathbf{a} < \mathbf{b}$.

Infinite levels: degree theory

- Let $\mathcal{D}_a$ denote the class of a-c.e. Turing degrees ($\mathcal{D}_\alpha$ makes sense for $\alpha < \omega^2$).
- The study of a-c.e. Turing degrees can be traced back to Ershov who showed that ω^2 is the first level where one can exhaust all Δ^0_2-degrees.

Theorem (Ershov, 1968)

(1) $\bigcup\{\mathcal{D}_a : |a|_o = \omega^2\} = \mathcal{D}(\leq \mathbf{0}')$.
(2) $\bigcup\{\mathcal{D}_a : |a|_o < \omega^2\} = \bigcup\{\mathcal{D}_\alpha : \alpha < \omega^2\} \subsetneq \mathcal{D}(\leq \mathbf{0}')$.

- Fix any notation a, the set $\bigcup\{\mathcal{D}_b : b <_o a\}$ will not exhaust all Δ^0_2-degrees.

Theorem (Selivanov, 1988)

For any $a \in \mathcal{O}$, $\bigcup\{\mathcal{D}_b : b <_o a\} \subsetneq \mathcal{D}(\leq \mathbf{0}')$.

- Though the finite levels of Ershov hierarchy were studied extensively, the infinite levels have not been explored sufficiently.

Main results

Motivated by Sacks Density Theorems, we show the following weak density theorem for infinite levels.

Theorem 1 (Liu and P., 2020)

For any $\mathbf{d} \in \mathcal{D}_\omega$ *and* $\mathbf{b} \in \mathcal{D}_n$ *with* $\mathbf{d} < \mathbf{b}$*, there is an* $\mathbf{a} \in \mathcal{D}_{\omega+1}$ *such that* $\mathbf{d} < \mathbf{a} < \mathbf{b}$.

Motivated by the d.c.e. Nondensity Theorem, we prove the following nondensity theorem for $(\omega + 1)$-c.e. degrees:

Theorem 2 (Liu and P., 2020)

For some notation a *with* $|a|_o = \omega^2$*, there is a* $\mathbf{d} \in \mathcal{D}_{\omega+1}$ *with* $\mathbf{d} < \mathbf{0}'$ *such that no* $\mathbf{a} \in \mathcal{D}_a$ *can have* $\mathbf{d} < \mathbf{a} < \mathbf{0}'$.

Elementary substructure problems

The Σ_1-elementary substructure problem for finite levels was completely settled.

Theorem (Cai, Shore and Slaman, 2012)

For any two natural numbers $n < m$, $\mathcal{D}_n \not\preceq_1 \mathcal{D}_m$.

For infinite levels, by Theorem 2 and the d.c.e. Nondensity Theorem.

Corollary

$\mathcal{D}_\omega \not\preceq_1 \mathcal{D}_{\omega+1}$.

An observation

By relativizing the downward density of $\mathcal{D}_n$ to $\mathcal{D}(\leq 0')$, we have: [1]

Proposition

For any $\mathbf{d} \in \mathcal{D}(\leq \mathbf{0}')$ and $\mathbf{b} \in \mathcal{D}_n$ with $\mathbf{d} < \mathbf{b}$, there is an $\mathbf{a} \in \mathcal{D}(\leq 0')$ such that $\mathbf{d} < \mathbf{a} < \mathbf{b}$.

Ershov showed that $\mathcal{D}(\leq \mathbf{0}') = \bigcup\{\mathcal{D}_a : |a|_o = \omega^2\}$. Then, in the above proposition, we can trivially find such $\mathbf{a}$ at level ω^2.

Question

Can we find such $\mathbf{a}$ at some level strictly below ω^2?

[1] Recall that Sacks showed that there is a $\mathbf{b} \in \mathcal{D}_\omega$ with $\mathbf{b} > \mathbf{0}$ such that no $\mathbf{a} \in \mathcal{D}(\leq \mathbf{0}')$ can have $\mathbf{0} < \mathbf{a} < \mathbf{b}$.

Summary

Given $\mathbf{d} \in \mathcal{D}(\leq \mathbf{0}')$ and $\mathbf{b} \in \mathcal{D}_n$ with $\mathbf{d} < \mathbf{b}$, find $\mathbf{a}$ such that $\mathbf{d} < \mathbf{a} < \mathbf{b}$.

1. When $\mathbf{d} \in \mathcal{D}_1$ and $\mathbf{b} \in \mathcal{D}_1$, then $\mathbf{a}$ can be found in $\mathcal{D}_1$. (Sacks, 1964).
2. When $\mathbf{d} \in \mathcal{D}_1$ and $\mathbf{b} \in \mathcal{D}_n$ $(n \geq 2)$, then $\mathbf{a}$ can be found in $\mathcal{D}_2$. (Cooper and Yi, 1995 for $n = 2$; Arslanov, Laforte and Slaman, 1998 for $n > 2$). [2]
3. When $\mathbf{d}$ is from $\mathcal{D}_2$ to $\mathcal{D}_\omega$, and $\mathbf{b} \in \mathcal{D}_n$ $(n \geq 1)$, then $\mathbf{a}$ can be found in $\mathcal{D}_{\omega+1}$. (Theorem 1). [3]
4. When $\mathbf{d} \in \mathcal{D}_{\omega+1}$ and $\mathbf{b} \in \mathcal{D}_n$, then $\mathbf{a}$ cannot necessarily be found at any level strictly below ω^2. (Theorem 2).

[2] This is the best level we could hope because Cooper and Yi showed that there exist $\mathbf{d} \in \mathcal{D}_1$ and $\mathbf{b} \in \mathcal{D}_2$ with $\mathbf{d} < \mathbf{b}$ such that no $\mathbf{a} \in \mathcal{D}_1$ can have $\mathbf{d} < \mathbf{a} < \mathbf{b}$.

[3] This is the best level we could hope because the d.c.e. Nondensity Theorem.

Normalizing notations: motivation

- We would like to extend previous results to the levels beyond ω^2.
- We will investigate these properties along a partial path T on $\mathcal{O}$.
- To generalize many results of lower levels to higher levels, we found that some nice properties of notations are needed.

Normalizing notations: motivation

We introduce the concept of normalizing notations, which is interesting in its own right.

Definition

A notation a is normalizing if for each $b <_o a$, we can effectively calculate $|b|_o$ from b, and calculate b from $|b|_o$.

In particular, given two notations $b_1, b_2 <_o a$ (assume $|b_1|_o + |b_2|_o < |a|_o$), we can effectively obtain a notation $b_3 <_o a$ such that $|b_3|_o = |b_1|_o + |b_2|_o$.

Remark

We have an effective addition $+_o$ on $\mathcal{O}$ such that for all $a, b \in \mathcal{O}$, $|a +_o b|_o = |a|_o + |b|_o$. However, $b_1 +_o b_2$ may not $<_o a$ in general.

Normalizing notations: ordinals below ε_0

Our concept of "normalizing" is only defined for those notations below level ε_0, this is because our method depends heavily on some Gödel numbering for ordinals below ε_0.

Definition

For ordinal α, its *Gödel number* $\ulcorner\alpha\urcorner = \ulcorner\omega^{\alpha_m} \cdot n_m + \cdots + \omega^{\alpha_0} \cdot n_0\urcorner$ $:=(\prod_{i\le m} p_{\ulcorner\alpha_i\urcorner}^{n_i}) - 1$, where p_i is the ith prime number starting with $p_0 := 2$. For natural number $x > 0$, its *corresponding ordinal* $o(x)=o((\prod_{i\le l} p_i^{n_i}) - 1)$ $:=\sum_{i\le l} \omega^{o(i)} \cdot n_i$. The sum is understood as the natural sum. Let $o(0) = 0$.

Definition

A notation a is *normalizing* if there are partial computable functions h_1, h_2 such that (1) if $b <_o a$, then $h_1(b)\downarrow= \ulcorner|b|_o\urcorner$ (2) if $\beta < |a|_o$, then $h_2(\ulcorner\beta\urcorner)\downarrow=$ the unique notation b such that $b <_o a$ and $|b|_o = \beta$.

Normalizing notations

Question

Do we have another method to give a similar concept of "normalizing" for all notations below level ω_1^{ck} *?*

Normalizing notations: construction

We will construct a short path T on $\mathcal{O}$ with length ε_0 such that for any ordinal $\alpha < \varepsilon_0$ (ε_0 is the least ordinal α such that $\alpha = \omega^\alpha$), there exists a normalizing notation $a \in T$ at level α.

Definition

Let $\alpha = \omega^{\alpha_m} + \cdots + \omega^{\alpha_1}$ $(\alpha_m \geq \cdots \geq \alpha_1)$ be a limit ordinal, we define a fixed *fundamental sequence* of α, $\alpha[0]$, $\alpha[1]$, $\alpha[2], \cdots$ as follows .

(i) If α is of the form $\beta + \omega^{\gamma+1}$, then $\alpha[n] = \beta + \omega^\gamma \cdot n$.

(ii) If α is of the form $\beta + \omega^\gamma$ and γ is a limit ordinal, then $\alpha[n] = \beta + \omega^{\gamma[n]}$.

Using the fundamental sequence, we can construct the short path $T \subseteq \mathcal{O}$ with $|T| = \varepsilon_0$. The construction is very straightforward.

Application: generalization of previous results

Let a_1, a_2 be normalizing notations.
On the one hand, [4]

Theorem

Suppose $|a_1|_o = \alpha + 1, |a_2|_o = \alpha \cdot \omega + 1$. *If* $\mathbf{d} \in \mathcal{D}_{a_1}$ *and* $\mathbf{d} < \mathbf{0}'$, *then there exists an* $\mathbf{a} \in \mathcal{D}_{a_2}$ *such that* $\mathbf{d} < \mathbf{a} < \mathbf{0}'$.

On the other hand, [5]

Theorem

Suppose $|a_1|_o = \alpha + 1, |a_2|_o = \alpha \cdot \omega$. *There exists a* $\mathbf{d} \in \mathcal{D}_{a_1}$ *with* $\mathbf{d} < \mathbf{0}'$ *such that there are no* $\mathbf{a} \in \mathcal{D}_{a_2}$ *with* $\mathbf{d} < \mathbf{a} < \mathbf{0}'$.

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[4] When $\alpha = 0$, this is Sacks Density Theorem (weak version). When $\alpha = 1$, this is Theorem 1 (weak version).

[5] When $\alpha = 1$, this is d.c.e. Nondensity Theorem. When $\alpha = \omega$, this is Theorem 2.

Application: minimal degrees

A limit ordinal α is called *indecomposable* if $\alpha = \omega^\delta$ for some ordinal δ (equivalently, there are no $\beta, \gamma < \alpha$ such that $\beta + \gamma = \alpha$).

Definition

If α is a decomposable ordinal and a a notation such that $|a|_o = \alpha$, then we say a is at decomposable level.

On the one hand,

Theorem

Suppose a is a normalizing notation at decomposable level, then there are no a-c.e. sets of minimal degree.

On the other hand,

Theorem

For any notation a (not necessary to be normalizing), there exists a minimal degree $\in \mathcal{D}(\leq \mathbf{0}') - \mathcal{D}_a$.

Thank you!