Some applications of the theory of Katětov order to ideal convergence

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1. Basics on ideals and ideal convergence
2. $\mathcal{I}$-closed sets and main problem
3. Basics on Katětov order

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e.g.,$\mathcal{I}_{\frac{1}{n}}$ is tall, while $\mathcal{I}_{\frac{1}{n}} \bigoplus \mathfrak{Fr}$ is not!(sum of two ideals: $\mathcal{I} \bigoplus \mathcal{J} = \{A \cup B : A \subseteq C, B \subseteq D, A \in \mathcal{I}, B \in \mathcal{J}\}$ is an ideal on $C \cup D$, where $\mathcal{I}$ is an ideal on C and $\mathcal{J}$ is an ideal on D and $C \cap D = \emptyset$)

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Note that a subsequence $\{x_n\}_{n \in A}$ is $\mathcal{I}$-convergent to x means: if $\{a_1, a_2, \cdots, a_k, \cdots\}$ is the increasing enumeration of A, then $\{k < \omega : x_{a_k} \notin U\} \in \mathcal{I}$ for every open neighborhood U of x.

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Problem (X.G. Zhou, L. Liu and S. Lin)

Whether every finite union of $\mathcal{I}$-closed subsets is $\mathcal{I}$-closed for every ideal $\mathcal{I}$?

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If $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$, then we say $\mathcal{I}, \mathcal{J}$ are Katětov equivalent, denoted by $\mathcal{I} =_K \mathcal{J}$.

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(i.e.,$\mathcal{I}|A =_K \mathcal{I}$ for every $\mathcal{I}$-positive set A)

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Theorem (D.Meza)

Every maximal ideal is K-uniform.

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KEY POINT. Let $\mathcal{I}$ be an ideal on ω and $A \subset \omega$. Denote $X(\mathcal{I}) = \omega \cup \{\omega\}$ in which each $n \in \omega$ is isolated and the topological base at ω is $\{\{\omega\} \cup F : F \in \mathcal{I}^*\}$.

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Proof sketch.

Let $\mathcal{I} = \mathcal{S} \bigoplus \mathcal{ED}$ and $X = X(\mathcal{I})$. Use $\mathcal{S} \not\leq_K \mathcal{I}$ and $\mathcal{ED} \not\leq_K \mathcal{I}$. □

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Possibly new cardinal invariants

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Let $\mathcal{P}$ be a class of topological spaces and $\mathcal{Q}$ be a class of ideals. Define $\mathfrak{fu}_{\chi}(\mathcal{P}, \mathcal{Q}) = \min\{\kappa$: there exists a space X with $\mathcal{P}$ and $\chi(X) = \kappa$ and a tall $\mathfrak{Fr} \subset \mathcal{I} \in \mathcal{Q}$ such that $\mathcal{I}$-closedness is not always preserved by finite unions in $X\}$.

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Is $\mathfrak{fu}_\chi(\emptyset, \emptyset) = \mathfrak{fu}_\chi(X(\mathcal{I}), F_\sigma)$*?*

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Thank you for your attention!

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